On the 2-systole of stretched enough positive scalar curvature metrics on S 2 x S 2

Thomas Richard

On the 2-systole of stretched enough positive scalar curvature metrics on $\mathbb{S}^2 \times \mathbb{S}^2$

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We use recent developments by Gromov and Zhu to derive an upper bound for the 2-systole of the homology class of $\mathbb{S}^2 \times \{*\}$ in a $\mathbb{S}^2 \times \mathbb{S}^2$ with a positive scalar curvature metric such that the set of spheres homologous to $\mathbb{S}^2 \times \{*\}$ is wide enough in some sense.

Recall that the systole of a compact Riemannian manifold (M^n, g) is the length of the shortest non contractible loop in (M^n, g). In the middle of the 20th century, Loewner and Pu proved sharp upper bounds on the systole of any metric on $\mathbb{T}^2$ or $\mathbb{RP}^2$ in term of its volume. These were vastly generalized when in the early 80s Gromov gave the similar (non sharp) bounds for n-dimensional essential manifolds. (See [Ber93] for the full story until 1993.)

The k-systole $\mathrm{sys}_k(g)$ of (M^n, g) is the infimum of the k-dimensional volume over all homologically non trivial k-cycles. In general, for $k \geq 2$, the k-systole of a manifold cannot be bounded by the volume alone. In particular, Katz and Suciu showed in [KS99] that one can find metrics on $\mathbb{S}^2 \times \mathbb{S}^2$ whose volume is 1 but whose 2-systole can be arbitrary large. (Similar examples in higher dimension where already known to Gromov, see again [Ber93].)

One way to circumvent this is to introduce the more subtle "stable systoles" (see again [Ber93]), another way is to try to introduce curvature restrictions. This second route was first considered in dimension 3 by Bray, Brendle and Neves for manifolds such as $\mathbb{S}^2 \times \mathbb{S}^1$ with positive scalar curvature metrics in [BBN10]. Zhu treated the case of $\mathbb{S}^2 \times \mathbb{T}^n$ $(n + 2 \leq 7)$ in [Zhu20], see Section 1 for a precise statement.

Here we will show how Zhu's result together with recent developments due to Gromov gives some progress in the case of $\mathbb{S}^2 \times \mathbb{S}^2$.

Let $\mathcal{S}_\ell$ be the set of embedded 2-spheres in $\mathbb{S}^2 \times \mathbb{S}^2$ which are in the same homology class as $\mathbb{S}^2 \times \{*\}$.

*LAMA, Univ Paris Est Creteil, Univ Gustave Eiffel, CNRS, F-94010, Créteil, France

Definition 0.1. *Let g be a Riemannian metric on $\mathbb{S}^2 \times \mathbb{S}^2$, the left stretch of g, denoted by* $\mathrm{stretch}_\ell(g)$ *is defined as*

$$\mathrm{stretch}_\ell(g) = \sup_{S_1, S_2 \in \mathcal{S}_\ell} d_g(S_1, S_2).$$

Example 0.2. For a product metric $g = g_1 \oplus g_2$ on $\mathbb{S}^2 \times \mathbb{S}^2$, the left stretch is $\mathrm{stretch}_\ell(g) = \mathrm{diam}(\mathbb{S}^2, g_2)$ and is achieved by $S_1 = \mathbb{S}^2 \times \{p_1\}$ and $S_2 = \mathbb{S}^2 \times \{p_2\}$ where $d_{g_2}(p_1, p_2) = \mathrm{diam}(\mathbb{S}^2, g_2)$.

Definition 0.3. *Let g be a Riemannian metric on $\mathbb{S}^2 \times \mathbb{S}^2$, the left 2-systole of g, denoted by* $\mathrm{sys}_{2,\ell}(g)$ *is defined as*

$$\mathrm{sys}_{2,\ell}(g) = \inf_{S \in \mathcal{S}_\ell} \mathrm{area}_g(S).$$

Example 0.4. For a product metric $g = g_1 \oplus g_2$ on $\mathbb{S}^2 \times \mathbb{S}^2$, the left 2-systole is $\mathrm{sys}_{2,\ell}(g) = \mathrm{area}(\mathbb{S}^2, g_1)$ and is achieved by any $S = \mathbb{S}^2 \times \{*\}$.

The theorem below is the main result of the paper. It says that positive scalar curvature metrics on $\mathbb{S}^2 \times \mathbb{S}^2$ with large left stretch cannot have large left 2-systole.

Theorem 0.5. *Let g be a metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with $\mathrm{Scal}_g \geq 4$.*
If $s = \mathrm{stretch}_\ell(\mathbb{S}^2 \times \mathbb{S}^2, g) > \frac{\sqrt{3}\pi}{2}$, then $\mathrm{sys}_\ell(\mathbb{S}^2 \times \mathbb{S}^2, g) \leq \frac{8\pi s^2}{4s^2 - 3\pi^2}$.

Remark 0.6. This estimate is asymptotically sharp when s goes to $+\infty$, as the example of the product fo two round spheres of radi $\frac{1}{\sqrt{k}}$ and $\frac{1}{\sqrt{2-k}}$ for $k \in [1, 2)$ shows.

Remark 0.7. Of course, if one denotes by $\mathcal{S}_r$ the set of embedded spheres homologous to $\{*\} \times \mathbb{S}^2$, we can define right stretch and systole as $\mathrm{stretch}_r(g) = \sup_{S_1, S_2 \in \mathcal{S}_r} d_g(S_1, S_2)$ and $\mathrm{sys}_{2,r}(g) = \inf_{S \in \mathcal{S}_r} \mathrm{area}_g(S)$ and get the same inequality between the right systole and the right stretch.

It is currently unknown wether the 2-systole is bounded from above on the set $\mathcal{M}_{\mathbb{S}^2 \times \mathbb{S}^2, \mathrm{Scal} \geq 4}$ of metrics on $\mathbb{S}^2 \times \mathbb{S}^2$ with scalar curvature greater than 4. Our result says that if there is no upper bound, there must be a sequence of metrics in $\mathcal{M}_{\mathbb{S}^2 \times \mathbb{S}^2, \mathrm{Scal} \geq 4}$ whose 2-systoles goes to infinity while their left and right stretches stay below $\frac{\sqrt{3}}{2}\pi$.

The rest of the paper is organized as follows: we first review some previously known results by Gromov and Zhu on manifolds with positive scalar curvature which will be used to prove Theorem 0.5, we then prove an elementary topological fact and we finally give the proof of the main theorem.

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1 Positive scalar curvature inequalities by Gromov and Zhu

We will need the following result by Jintian Zhu, which was already alluded to in the introduction.

Theorem 1.1 ([Zhu20], Theorem 1.2)**.** *Let* (M^n, g) *be a closed manifold of dimension at most seven such that:*

- $\mathrm{Scal}_g \geq 2$*;*
- *there exists a map* $F : M^n \to \mathbb{S}^2 \times \mathbb{T}^{n-2}$ *with non-vanishing degree.*

Then $\mathrm{sys}_2(g) \leq 4\pi$*. More precisely one can find an embedded 2-sphere* S *such that* $F_*([S]) = [\mathbb{S}^2 \times \{*\}]$ *and whose area is less than* 4π*.*

The proof uses the second variation formula of the area functional for stable minimal hypersurfaces and a repeated symmetrization construction to reduce the problem to $\mathbb{T}^{n-2}$-invariant metrics on $\Sigma^2 \times \mathbb{T}^{n-2}$. This method of proof was pioneered by Fischer-Colbrie and Schoen in dimension 3 and used in dimensions at most seven by Gromov and Lawson in [GL83]. The author also studies the equality case.

Remark 1.2. The proof in [Zhu20] shows that if one starts with an $\mathbb{S}^1$-invariant metric on $M = M' \times \mathbb{S}^1$ satisfying the hypothesis of Theorem 1.1, then one can find a 2-sphere in M' whose area is at most 4π.

In the Spring of 2019, the author was lucky enough to attend the lectures "Old, New and Unknown around Scalar Curvature" given by Gromov at IHES. There, the author learned about the following result by Gromov which is also proved using the Fischer-Colbrie–Schoen symmetrization process:

Theorem 1.3 ([Gro18], p. 2)**.** *Let* $2 \leq n \leq 7$*,* $M = [-1, 1] \times \mathbb{T}^{n-1}$ *and* $\partial_\pm M = \{\pm 1\} \times \mathbb{T}^{n-1}$*. Let* g *be a Riemannian metric on* M *such that* $\mathrm{Scal}_g \geq n(n-1)$*. Then*

$$d_g\big(\partial_- M, \partial_+ M\big) \leq \frac{2\pi}{n}.$$

In his Spring 2019 lectures, Gromov gave a new proof of the previous theorem. Instead of using the second variation of the $(n-1)$-dimensional volume as in [Gro18], a twisted functional is considered. Given a density $\mu : M^n \to \mathbb{R}$, the μ-area functional maps an open set $U \subset M$ to $\mathcal{V}_{n-1}(\partial U) - \int_U \mu$, where $\mathcal{V}_{n-1}$ denotes the $(n-1)$-dimensional volume. Using a well chosen density μ, Gromov proved the following theorem, which is a generalisation of the theorem above.

Theorem 1.4 ([Gro19], Section 5.3)**.** *Let* (M^n, g) $(n \leq 7)$ *be a compact* n*-manifold with two boundary components* $\partial_- M$ *and* $\partial_+ M$*. Assume that*

$$\mathrm{Scal}_g \geq \frac{4(n-1)\pi^2}{n\, d_g(\partial_- M, \partial_+ M)^2} + \delta$$

for some $\delta > 0$*. Then there exists*

- *a connected hypersurface Σ which separates $\partial_- M$ and $\partial_+ M$,*
- *a positive function $u : \Sigma \to \mathbb{R}$,*

such that the metric $h = g|_\Sigma + u^2 dt^2$ on $\Sigma \times \mathbb{R}$ has $\mathrm{Scal}_h \geq \delta$.

2 Topological preliminaries

Before proving the main theorem, we establish a topological preliminary. Let $M = \mathbb{S}^2 \times \mathbb{S}^2$.

The idea is that if we remove two disjoint spheres S_1 and S_2 in $\mathcal{S}_\ell$ from M, we should be in a situation which is very similar to removing $\mathbb{S}^2 \times \{n\}$ and $\mathbb{S}^2 \times \{s\}$ from $\mathbb{S}^2 \times \mathbb{S}^2$, where n and s are the north and south poles of the right factor. Thus $\tilde{M} = M \backslash (S_1 \cup S_2)$ should look like

$$\mathbb{S}^2 \times \mathbb{S}^2 \backslash \left(\mathbb{S}^2 \times \{n\} \cup \mathbb{S}^2 \times \{s\}\right) = \mathbb{S}^2 \times \left(\mathbb{S}^2 \backslash \{n, s\}\right) \simeq \mathbb{S}^2 \times \mathbb{S}^1 \times (-1, 1).$$

It may well be that $M \backslash (S_1 \cup S_2)$ is always diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1 \times (-1, 1)$ but the tools at the hands of the author didn't provide a simple proof of this fact. Anyway, it is enough for our purpose to prove :

Proposition 2.1. *Let S_1 and S_2 be two disjoint spheres in $\mathcal{S}_\ell$. Let $\Sigma \subset M$ be a connected hypersurface such that*

- *Σ is disjoint from S_1 and S_2,*
- *Σ separates S_1 and S_2.*

Then there exists a non-zero degree map $\Sigma \to \mathbb{S}^2 \times \mathbb{S}^1$.

This map will be the restriction of a map $F : \tilde{M} = M \backslash (S_1 \cup S_2) \to \mathbb{S}^2 \times \mathbb{S}^1$.

We will first show :

Lemma 2.2. *Any $S \in \mathcal{S}_\ell$ has a trivial normal bundle.*

Proof. Since the tangent bundle TS and the normal bundle NS of S are orientable real plane bundles, we can see them as complex line bundles. To prove that NS is trivial as a real bundle, it is enough to show that its first Chern class vanishes.

Let ι denote the inclusion $\iota : S \to M$. Then $\iota^* TM = TS \oplus NS$. Let ω_ℓ and ω_r be the two standard generators of $H^2(M)$ coming from the left and right factors of $M = \mathbb{S}^2 \times \mathbb{S}^2$, then the total Chern class of TM is given by $c(TM) = (1 + 2\omega_\ell)(1 + 2\omega_r)$ by the Whitney product formula (see for instance [RB95], p. 270).

Now $c(\iota^* TM) = \iota^* c(TM) = 1 + 2\omega_\ell$ since $\iota^*(\omega_\ell) = 0$. Moreover, $c(TS) = (1 + \chi(S)\omega_\ell)$ and $c(\iota^* TM) = c(TS \oplus NS) = c(TS)c(NS)$. Hence $c(NS) = 1 + (2 - \chi(S))\omega_\ell = 1 + 2\gamma\omega_\ell$ if γ is the genus of S.

Since S has genus 0, $c(NS) = 1$ and NS is trivial. $\square$

For $\varepsilon > 0$, we denote by S_i^ε the tubular neighborhood around S_i. We will choose ε small enough so that S_1^ε and S_2^ε are disjoint regular neighborhoods.

Lemma 2.3. *The map $F_2 : \tilde{M} \subset \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{S}^2 \times \{*\}$ is surjective in homology.*

Proof. Consider a 3-cycle C in M such that $\partial C = S_1 - S_2$. By using an fine enough triangulation, we can decompose $C = C_1 + \tilde{C} + C_2$, where $\tilde{C}$ is a 3-cycle in $\tilde{M}$ and C_i is a 3-cycle in S_i^ε. Now, $S_1' = \partial C_1 - S_1$ is a 2-cycle in $\tilde{M}$ homologous to S_1, hence $\pi(F_2)_*([S_1']) = [\mathbb{S}^2 \times \{*\}]$. □

Set $S = S_1 \cup S_2$, $S^\varepsilon = S_1^\varepsilon \cup S_2^\varepsilon$ and $\tilde{S}^\varepsilon = S^\varepsilon \backslash S$.

Lemma 2.4. *$H^1(\tilde{M}, \mathbb{Z}) = \mathbb{Z}$.*

Remark 2.5. The restriction to spheres in Definition 0.3 comes from here. It would be more satisfying to allow any surface homologous to $\mathbb{S}^2 \times \{*\}$ but we would loose Lemma 2.2 which would in turn invalidate this computation of $H^1(\tilde{M}, \mathbb{Z})$. A more careful computation shows that if S_1 and S_2 both have non zero genus then $H^1(\tilde{M}, \mathbb{Z}) = 0$.

Proof. We will use the Mayer-Vietoris exact sequence for cohomology associated with the decomposition $M = S^\varepsilon \cup \tilde{M}$. Note that S^ε is homotopy equivalent to S and since the S_1 and S_2 have trivial normal bundle, $\tilde{S}^\varepsilon$ is homotopy equivalent to $S \times \mathbb{S}^1$.

The Mayer Vietoris sequence gives

$$\cdots \to H^1(M) \to H^1(S^\varepsilon) \oplus H^1(\tilde{M}) \to H^1(\tilde{S}^\varepsilon) \to H^2(M) \to \cdots$$

Thus, we have

$$\cdots \to 0 \to 0 \oplus H^1(\tilde{M}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \cdots$$

where the map $H^1(\tilde{S}^\varepsilon) \to H^2(M)$ is given by $(x, y) \mapsto (x - y, 0)$. Hence $H^1(\tilde{M})$ is isomorphic to $\mathbb{Z}$. □

Since $H^1(\tilde{M}, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1(\tilde{M}, \mathbb{R}) \simeq \mathbb{R}$ and we can find a closed 1-form α on $\tilde{M}$ which is not exact. Let Γ be the image in $\mathbb{R}$ of the abelian group morphism : $c \in H_1(\tilde{M}, \mathbb{Z}) \mapsto \int_c \alpha \in \mathbb{R}$. By a computation similar to the proof of the previous lemma, $H_1(\tilde{M}, \mathbb{Z}) = \mathbb{Z}$. Hence Γ is a discrete subgroup of $\mathbb{R}$. The classical period map construction gives :

Lemma 2.6. *Let $x_0 \in \tilde{M}$. For any curve c from x_0 to x, $\int_c \alpha$ is well defined as an element $\mathbb{R}/\Gamma$. This defines a map $F_1 : \tilde{M} \to \mathbb{R}/\Gamma$ such that the induced map $H_1(\tilde{M}) \to H_1(\mathbb{R}/\Gamma)$ is an isomorphism.*

$F = (F_2, F_1) : \tilde{M} \to \mathbb{S}^2 \times \mathbb{S}^1$ is the map we are looking for.

Proof of Proposition 2.1. First note that any connected hypersurface $\Sigma \subset M$ which separates S_1 and S_2 will be homologous to ∂S_1^ε for ε small enough. Thus it is enough to show that the image of $[\partial S_1^\varepsilon]$ under F is not zero in $H_3(\mathbb{S}^2 \times \mathbb{S}^1) = H_2(\mathbb{S}^2) \oplus H_1(\mathbb{S}^1)$, which is routinely deduced from the proofs of Lemmas 2.3 and 2.4.

□

3 Proof of the main theorem

We are now ready to prove Theorem 0.5.

Proof of Theorem 0.5. Our assumption is that $s = \mathrm{stretch}_\ell(\mathbb{S}^2 \times \mathbb{S}^2, g) > \frac{\sqrt{3}\pi}{2}$. Thus, for any $\varepsilon > 0$ we can find two spheres S_1 and S_2 in $\mathcal{S}_\ell$ such that $d_g(S_1, S_2) = s - \varepsilon$. Assume that $\varepsilon > 0$ is such that

- $s - 3\varepsilon > \frac{\sqrt{3}\pi}{2}$;
- the tubular neighborhoods S_1^ε and S_2^ε of radius ε around S_1 and S_2 are disjoint and regular.

Then if we set $\tilde{M} = M \backslash (S_1^\varepsilon \cup S_2^\varepsilon)$, its two boundary components $\partial_- \tilde{M} = \partial S_1^\varepsilon$ and $\partial_+ \tilde{M} = \partial S_2^\varepsilon$ satisfy $d_g(\partial_- \tilde{M}, \partial_+ \tilde{M}) = s - 3\varepsilon$.

Set $\delta = 4 - \frac{3\pi^2}{(s-3\varepsilon)^2}$ and note that $\delta > 0$. Then we have:

$$\mathrm{Scal}_g \geq 4 = \frac{3\pi^2}{d_g(\partial_- \tilde{M}, \partial_+ \tilde{M})^2} + \delta.$$

This is exactly what we need to apply Gromov's Theorem 1.4. Hence we get an hypersurface $\Sigma \subset \tilde{M}$ which separates S_1 and S_2 such that $\Sigma \times \mathbb{S}^1$ admits a metric of the form $h = g_\Sigma + u^2 dt^2$ for some $u : M \to \mathbb{R}$ such that $\mathrm{Scal}_h \geq \delta$.

Moreover, by Proposition 2.1, there exists a non-zero degree map $\Sigma \to \mathbb{S}^2 \times \mathbb{S}^1$. Hence there is a non-zero degree map $\Sigma \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{T}^2$ and we can apply Zhu's Theorem 1.1 and Remark 1.2 to show that one can find an embedded 2-sphere S in Σ whose area is at most $\frac{8\pi}{\delta} = \frac{8\pi(s-3\varepsilon)^2}{4(s-3\varepsilon)^2 - 3\pi^2}$.

Since Σ is isometrically embedded in M, S embeds in M with area at most $\frac{8\pi(s-3\varepsilon)^2}{4(s-3\varepsilon)^2 - 3\pi^2}$. Moreover, by the properties of the map F built in section 2, the surface S of M belongs to $\mathcal{S}_\ell$. Since $\varepsilon > 0$ can be as small as one wants we get the results. ☐

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