Small Two Spheres in Positive Scalar Curvature, Using Minimal Hypersurfaces

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Abstract. In this survey, we review some results on the area of topologically non trivial two spheres in manifolds with positive scalar curvature of dimension at most 7. The main tool to get those results is the use of stable minimal hypersurfaces, and we give a short exposition of the usual ways to take advantage of these in the presence of positive scalar: second variation, conformal method, Fischer-Colbrie–Schoen symmetrization and μ-bubbles. The last sections lists some open problems in the area.

1. Introduction

The study of positively curved Riemannian manifolds saw great developments in the second half of the 20th century. In the case of positive sectional and Ricci curvature, the quantitative theorems of comparison geometry such as Toponogov's theorem, Myers theorem or the Bishop-Gromov inequality, were pivotal tools in this development, see for instance the classic ([CE75]).

At the same time, the study of manifolds with positive scalar curvature took a more qualitative approach based on one side on the properties of the Dirac operator on spin manifolds ([Lic63], [GL83]) and on the other side on the study of stable minimal hypersurfaces ([SY79]).

Most of the results were topological in nature, the most celebrated one being the impossibility to endow the torus $\mathbb{T}^n$ with a positive scalar curvature metric. This theorem was proved in dimension at most seven by Schoen and Yau in the late 80's using minimal hypersurfaces and by Gromov and Lawson in any dimension soon afterwards using Dirac operators methods.

In these same papers, it was noted that one can build new positive scalar curvature manifolds by taking connected sums (or higher codimension surgeries). This explains a bit why finding metric invariants which can be controlled by scalar curvature is difficult: the sought for invariants should be insensitive to connected sums with long or big manifolds.

The goal of this survey is to show how in certain contexts positive scalar curvature can be used to control the area of the smallest topologically non-trivial 2-spheres in a positive scalar curvature manifold. We illustrate in particular how hypersurfaces, which are codimension 1, can be used to get information about higher codimension submanifolds.

Let g be a smooth metric on $\mathbb{S}^2$. The classical Gauss-Bonnet formula yields

$$\int_{\mathbb{S}^2} R_g \,\mathrm{d}\sigma_g = 8\pi,$$

where R_g and $\mathrm{d}\sigma_g$ are scalar curvature and area element of $(\mathbb{S}^2, g)$ respectively. As a corollary, $(\mathbb{S}^2, g)$ has area no greater than 4π if the scalar curvature R_g is no less than 2. Despite its simplicity, this fact suggests the following principle: *Large positive scalar curvature favors the existence of* 2*-spheres with small area*, as we will see here in more general situations.

When looking for small 2-spheres in a given manifold, it is a good idea to focus on a particular class of 2-spheres and look for the infimum of the area functional restricted to this class. For a Riemannian manifold (M, g) with non-trivial second homotopy group, we take the following *homotopical* 2*-systole*

$$\mathrm{sys}_2(M,g) = \inf\left\{\mathrm{area}(\mathbb{S}^2, i^*g) \,\middle|\, \begin{array}{c} i:\mathbb{S}^2\to M \text{ smooth such that} \\ [i] \text{ homotopically nontrivial} \end{array}\right\}$$

raised up by Bray, Brendle and Neves in their work [BBN10], where they proved

Theorem 1.1. *If (M, g) is a closed Riemannian* 3*-manifold with non-trivial second homotopy group whose scalar curvature is no less than* 2*, then it holds* $\mathrm{sys}_2(M, g) \le 4\pi$*. Moreover, the equality holds if and only if the universal covering of (M, g) is isometric to the product manifold $(\mathbb{S}^2 \times \mathbb{R}, g_{round} + \mathrm{d}t^2)$, where g_{round} denotes the standard round metric on* 2*-sphere.*

In general, no similar result can be expected for closed Riemannian manifolds with dimension greater than 3. For $n \ge 5$, one can simply take $\mathbb{S}^{n-2}(r_1) \times \mathbb{S}^2(r_2)$ with suitably chosen radius r_1 and r_2 as counterexamples. As a consequence, more requirements are necessary to guarantee the existence of small two spheres from large positive scalar curvature in dimensions greater than 3. In this direction, the second named author investigated the class of closed n-manifolds that admits a smooth map to $\mathbb{S}^2 \times \mathbb{T}^{n-2}$ with non-zero degree. In dimension n no greater than 7, he showed the following result in [Zhu20a].

Theorem 1.2. *Let (M, g) be a closed oriented Riemannian n-manifold with scalar curvature R_g no less than* 2*, which also admits a smooth map $f : M \to \mathbb{S}^2 \times \mathbb{T}^{n-2}$ with non-zero degree. Then M has non-trivial second homotopy group and it holds* $\mathrm{sys}_2(M, g) \le 4\pi$*. Furthermore, the equality holds if and only if the universal covering of (M, g) is isometric to the product manifold $(\mathbb{S}^2 \times \mathbb{R}^{n-2}, g_{round} + g_{euc})$, where g_{round} is the standard round metric on $\mathbb{S}^2$ and g_{euc} denotes the standard Euclidean metric on $\mathbb{R}^{n-2}$.*

For compact manifolds with positive scalar curvature, the only other 2-systole estimate has been proved by the second author in [Ric20] in the case of positive scalar curvature metrics on $\mathbb{S}^2 \times \mathbb{S}^2$ wtih an additional ”stretching” assumption. Namely

Theorem 1.3. *Let g be a smooth metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with R_g no less than* 4 *such that the left stretch*

$$s = \sup_{S_1, S_2 \in \mathcal{S}_l} \mathrm{dist}_g(S_1, S_2) > \frac{\sqrt{3}\pi}{2},$$

where $\mathcal{S}_l$ denotes the set of all embedded surfaces representing the homology class $[\mathbb{S}^2 \times \{*\}]$. Then it holds

$$\mathrm{sys}_2(\mathbb{S}^2 \times \mathbb{S}^2, g) \le \frac{8\pi s^2}{4s^2 - 3\pi^2}.$$

As for complete open Riemannian manifolds with positive scalar curvature, the second named author also showed some optimal 2-systole estimates in [Zhu20b]. First he made a generalization to Theorem 1.1 as following

Theorem 1.4. *Let (M, g) be a complete open Riemannian manifold with non-trivial second homotopy group. If the scalar curvature of (M, g) is no less than 2, then it holds $\mathrm{sys}_2(M, g) \leq 4\pi$, where the equality holds if and only if the universal covering of (M, g) is isometric to the product manifold $(\mathbb{S}^2 \times \mathbb{R}, g_{round} + \mathrm{d}t^2)$.*

In the case where the dimension n is no greater than 7, he proved

Theorem 1.5. *Let (M, g) be a complete, oriented, open Riemannian n-manifold with scalar curvature R_g no less than 2 for $n \leq 7$, which admits a smooth proper map $f : M \to \mathbb{S}^2 \times \mathbb{T}^{n-3} \times \mathbb{R}$. Then M has non-trivial second homotopy group and it holds $\mathrm{sys}_2(M, g) \leq 4\pi$, where the equality holds if and only if the universal covering of (M,g) is isometric to the product manifold $(\mathbb{S}^2 \times \mathbb{R}^{n-2}, g_{round} + g_{euc})$.*

The rest of this survey is organised as follows: in section 2 we give a brief overview of how stable minimal surfaces in 3-manifolds have been used to get topological and quantitative information from a positive scalar curvature assumption. In section 3 we sketch how the conformal method was used to get topological information on positive scalar curvature manifolds in dimension 4 to 7 and why it failed to prove quantitative statements. In section 4 we introduce the Fischer-Colbrie–Schoen symmetrization process and show how it implies Theorem 1.2. Section 5 deals with applications of Gromov's μ-bubble to these kinds of questions. Theorems 1.3, 1.4 and 1.5 are presented there in more details. The final section gathers open questions in this subject.

2. Stable hypersurfaces and stable 2-spheres in 3-manifolds

2.1. **Variation formulas, and the Schoen–Yau trick.** Let (M^n, g) be a complete Riemannian manifold of dimension at most seven, $\Sigma^{n-1} \subset M^n$ be a two-sided minimal hypersurface, and let ν be a unit normal field along Σ. Consider a smooth family of hypersurfaces $t \mapsto \Sigma_t$ such that:

- $\Sigma_0 = \Sigma$.
- $\frac{d}{dt}|_{t=0}\Sigma_t = f\nu$ for some smooth function $f : \Sigma \to \mathbb{R}$.

In this setting, the second variation of the $t \mapsto \mathrm{area}_g\, \Sigma_t$ is given by :

$$\frac{d^2}{dt^2}_{|t=0} \mathrm{area}(\Sigma_t) = \int_\Sigma |\nabla f|^2 - \left(|\,\mathrm{A}\,|^2 + \mathrm{Ric}_M(\nu, \nu)\right) f^2 d\sigma$$

where A is the second fundamental form of Σ.

Definition 2.1. Σ is said to be stable if for all $f \in C^\infty(\Sigma)$:

$$\int_\Sigma |\nabla f|^2 - \left(|\,\mathrm{A}\,|^2 + \mathrm{Ric}_M(\nu, \nu)\right) f^2 d\sigma \geq 0$$

One way to get a stable hypersurface is when Σ is built by looking for a minimizer of the area functional in some non trivial homology class. Federer's geometric measure theory ensures the existence of a minimizing rectifiable current if M is compact, while Almgren's regularity theory gives that this currents comes from a smooth hypersurface if M has dimension at most 7. (See for instance [Mor16] and the references therein.)

At first glance, it is unclear how stable minimal hypersurfaces can be of any use in the study of Riemannian manifolds with positive scalar curvature. The crucial observation was made by Schoen and Yau in the late seventies ([SY79]):

Lemma 2.2. *If $\Sigma \subset M$ is minimal,then :*

$$2\,\mathrm{Ric}_M(\nu,\nu) = R_M - R_\Sigma - |\,\mathrm{A}\,|^2$$

Proof. The proof is obtained by tracing the Gauss equations $\mathrm{Rm}_\Sigma = \mathrm{Rm}_M + \mathrm{A}\wedge\mathrm{A}$ twice. □

Hence the stability condition can be restated as follows: if Σ is a two-sided stable minimal hypersurface then for any function $f : \Sigma \to \mathbb{R}$ we have :

$$\int_\Sigma |\nabla f|^2 - \tfrac{1}{2}\left(R_M - R_\Sigma + |\,\mathrm{A}\,|^2\right) f^2 d\sigma \geq 0.$$

2.2. **No positive scalar curvature metric on $\mathbb{T}^3$.** In dimension 3, the second variation formula shows that stable minimal surfaces must be spheres:

Proposition 2.3. *Let (M^3, g) be an orientable 3-manifold with $R_g > 0$ and let $\Sigma^2 \subset M^3$ be a stable two-sided minimal surface, then Σ^2 is diffeomorphic to $\mathbb{S}^2$.*

Proof. We know that if Σ is stable and 2-sided then

$$\int_\Sigma |\nabla f|^2 - \tfrac{1}{2}\left(R_M - R_\Sigma + |\,\mathrm{A}\,|^2\right) f^2 d\sigma \geq 0$$

for any function $f : \Sigma \to \mathbb{R}$. Plugging the constant function equal to 1 in this inequality, we get :

$$\int_\Sigma R_\Sigma d\sigma \geq \int_\Sigma R_M + |\,\mathrm{A}\,|^2 d\sigma > 0.$$

Since the scalar curvature in dimension 2 is twice the Gauss curvature, this shows using Gauss-Bonnet that $\chi(\Sigma^2) > 0$. Hence, since Σ^2 is orientable it is diffeomorphic to $\mathbb{S}^2$. □

Using this simple observation, Schoen and Yau showed the Geroch conjecture in dimension 3 [SY78]:

Theorem 2.4. *$\mathbb{T}^3$ has no metric with positive scalar curvature.*

Proof. Using the proposition above, it is enough to build a stable orientable minimal surface in $\mathbb{T}^3$ which is not a sphere.

One way to do it is to consider $\mathcal{X}$ the homology class of $\mathbb{T}^2 \times \{*\} \subset \mathbb{T}^3$. Using geometric measure theory there is a stable minimal surface Σ^2 in $\mathcal{X}$.

To see that Σ^2 is not a sphere, consider the projection $\pi : \mathbb{T}^3 \to \mathbb{T}^2 \times \{*\}$, and let α_1 and α_2 be the two standard generators of the de Rahm cohomology group $H^1(\mathbb{T}^2)$. Since Σ and $\mathbb{T}^2 \times \{*\}$ are homologous: $\int_{\Sigma^2} \pi^*(\alpha_1 \wedge \alpha_2) = \int_{\mathbb{T}^2\times\{*\}} \alpha_1 \wedge \alpha_2 = 1$, which implies that $\pi : \Sigma^2 \to \mathbb{T}^2 \times \{*\}$ has non zero degree and thus Σ^2 is not a sphere. □

2.3. **Bray–Brendle–Neves result on small spheres in p.s.c 3-manifolds.** Surprisingly, it took thirty years for geometers to realize that the second variation not only ruled out positive scalar curvature metrics on $\mathbb{T}^3$ but also gave some quantitative information on positive scalar curvature metrics on 3-manifolds. This was proved by Bray, Brendle and Neves:

Theorem 2.5. *Let (M^3, g) be a compact 3-manifold with $R_g \geq 2$ and $\pi_2(M^3) \neq \{0\}$. Then $\mathrm{sys}_2(M, g) \leq 4\pi$, morever if equality is achieved then the universal cover of (M^3, g) is isometric $\mathbb{S}^2 \times \mathbb{R}$ with the product metric of constant scalar curvature 2.*

Proof. We will only prove the inequality in this section. For the rigidity part, see section 4.2.

Let Σ^2 be a minimizer of the area among all non contractible 2-spheres in M^3. Such a minimzer can be found thanks to results by Meeks and Yau ([MY80]). We now write the stability inequality :

$$\int_\Sigma R_\Sigma d\sigma \geq \int_\Sigma R_M + |\,\mathrm{A}\,|^2 d\sigma.$$

By the Gauss-Bonnet formula, we have that $\int_\Sigma R_\Sigma d\sigma = 8\pi$. Hence, since $R_M \geq 2$, we get:

$$2\,\mathrm{area}\,\Sigma \leq \int_\Sigma R_M \leq \int_\Sigma R_\Sigma d\sigma = 8\pi$$

and $\mathrm{area}\,\Sigma \leq 4\pi$. □

3. Higher dimensions: the conformal method and its failure to prove quantitative results

In this section we will quickly review the progress made in the use of hypersurfaces in the study of positive scalar curvature manifold of dimension between 4 and 7, and expalin why it cannot be used to get quantitative informations.

Schoen and Yau proved in [SY79] :

Theorem 3.1. *Let (M^n, g) be a manifold such that:*

- $n \leq 7$.
- *There exsits a non-zero degree map $M^n \to \mathbb{T}^n$.*

then (M^n, g) admits non metric with positive scalar curvature.

This in particular implies the Geroch conjecture in dimension at most 7.

First part of the proof. The proof goes by induction on n. The cases $n = 2$ and $n = 3$ are handled by the previous section.

The proof of the induction step starts as in section 2.2. Assume M^{n+1} satisfies the assumptions of the theorem and has positive scalar curvature. We will show that this contradicts the theorem in dimension n.

Consider the projection $\pi : M^{n+1} \to \mathbb{T}^{n+1} \to \mathbb{T}^1$ of M^n onto the last factor of $\mathbb{T}^{n+1}$. Let $\mathcal{X}$ be the homology class of generic fiber of π. Then for any smooth hypersurface $\Sigma \in \mathcal{X}$ the projection $\Sigma^n \subset M^{n+1} \to \mathbb{T}^{n+1} \to \mathbb{T}^n$ on the first n factors of $\mathbb{T}^{n+1}$ has non zero degree. Let us consider a minimizer Σ^n of the area in $\mathcal{X}$ which will be smooth since $n + 1 \leq 7$. If we can prove that Σ^n admits a positive scalar curvature metric we're done.

Arguing as in dimension 3 by plugging the constant function in the second variation formula gives nothing more than $\int_\Sigma R_\Sigma d\sigma > 0$: the scalar curvature is positive *in average*. This is not enough! □

To take advantage of the full second variation (and not just variations along the unit normal), we need to make a small detour in conformal land, see for instance [Bes87]. The conformal Laplacian L_Σ on Σ is the operator on functions given by:

$$L_\Sigma f = -4\tfrac{n-1}{n-2}\Delta_\Sigma f + R_\Sigma f.$$

It is tied to conformal geometry by the fact that the scalar curvature of the metric $f^{\frac{4}{n-2}} g_\Sigma$ is given by $f^{\frac{n+2}{n-2}} L_\Sigma f$.

As a consequence of this formula, if L_Σ is positive we can can use the first eigenfunction f_1 of L_Σ (which will be positive) as a conformal factor to build a metric $f_1^{\frac{4}{n-2}} g_\Sigma$ whose scalar curvature is $\lambda_1 f_1^{\frac{2n}{n-2}}$ where $\lambda_1 > 0$ is the first eigenvalue of L_Σ, which is positive.

End of the proof. Now if Σ is a stable minimal hypersurface in manifold with positive curvature, we have that for any $f : M \to \mathbb{R}$:

$$\int_\Sigma |\nabla f|^2 + \tfrac{1}{2} R_\Sigma f^2 d\sigma > 0$$

Hence as operators we have $\frac{1}{2} R_\Sigma > \Delta_\Sigma$. Thus we can compute:

$$L_\Sigma = -4\tfrac{n-1}{n-2}\Delta_\Sigma + R_\Sigma > \left(-4\tfrac{n-1}{n-2} + 2\right)\Delta_\Sigma = -\frac{2n}{n-2}\Delta_\Sigma$$

which shows that the conformal Laplaction L_Σ is positive. Hence Σ admits a metric with positive scalar curvature and we have completed the induction step. □

If we want to use this method to get quantitative results, we first assume that $R_M \geq \sigma > 0$. This will give that the conformal laplacian satisfies $L_\Sigma \geq \sigma$ and thus its first eigenvalue λ_1 satisfies $\lambda_1 \geq \sigma$.

Unfortunately, this only gives that the scalar curvature of Σ with the metric $f_1^{\frac{4}{n-2}} g_\Sigma$ is bigger than $\sigma f_1^{\frac{2n}{n-2}}$. Since f_1 is only defined up to a positive factor this lower bound cannot be used.

Moreover, if we would like to prove a metric inequality with this dimension descent method, we would need to relate metric invariants of the conformally deformed $(\Sigma, f_1^{\frac{4}{n-2}} g_\Sigma)$ to those of (Σ, g_Σ) in order to (hopefully) learn something about the metric invariants of (M, g). This seems (at best) difficult.

4. Quantitative implications of uniformly positive scalar curvature

4.1. **Fischer-Colbrie–Schoen symmetrization.** In the last section we saw how the conformal method pushed the use of minimal hypersurfaces beyond dimension 3 to give topological information on positive scalar curvature manifolds and how it failed to give quantitative information.

Fischer-Colbrie and Schoen found another method to deal with stable minimal hypersurfaces in [FS80]. Their original goal was to embed a complete Riemannian plane (P, g) associated with a smooth positive function u on P such that $\Delta_g u - K_g u = 0$ inside a complete scalar-flat Riemannian 3-manifold as a stable minimal surface. Starting with such a function u, they constructed a new metric $\tilde{g} = g + u^2 \mathrm{d}t^2$ on $P \times \mathbb{T}^1$ and a direct computation shows

$$R_{\tilde{g}} = R_g - \frac{2\Delta_g u}{u} = 0.$$

The above construction is now called *Fischer-Colbrie–Schoen symmetrization.*

As illustration of this method in higher dimension, one can given another proof for the Geroch conjecture using Fischer-Colbrie–Schoen symmetrization. To ensure smoothness of area minimizing hypersurfaces, we work in dimension at most 7. We start with an orientable closed Riemannian manifold (M^n, g) with positive scalar curvature, which also admits a continuous map $f : M \to \mathbb{T}^n$ with non-zero degree map. From geometric measure theory we can find an area-minimizing minimal hypersurface Σ in M such that Σ also admits a continuous map $f' : \Sigma \to \mathbb{T}^{n-1}$. In particular, Σ is stable and the operator $-\Delta_g - \frac{1}{2}\left(R_g - R_\Sigma + |\,\mathrm{A}\,|^2\right)$ is nonnegative. Hence its first eigenfunction u is smooth, positive, and satisfies:

$$\Delta_\Sigma u + \frac{1}{2}\left(R_g - R_\Sigma + |A|^2\right) u \leq 0.$$

From Fischer-Colbrie-Schoen symmetrization, we are able to construct a new metric $\tilde{g} = g + u^2 \mathrm{d}t^2$ on $\tilde{M} = M \times \mathbb{T}^1$ such that

$$R_{\tilde{g}} = R_\Sigma - \frac{2\Delta_\Sigma u}{u} \geq R_g + |A|^2 > 0.$$

Notice that we have constructed a new Riemannian manifold $(\tilde{M}, \tilde{g})$ with positive scalar curvature admitting a continuous map $\tilde{f} : \tilde{M} \to \mathbb{T}^n$ with non-zero degree map. What's more, $\tilde{M}$ becomes simpler than M since it is $\mathbb{T}^1$-symmetric. Not surprisingly, iterating this process, we will obtain a warped product metric

$$u_1(t_1)^2 \mathrm{d}t_1^2 + u_2(t_1)^2 \mathrm{d}t_2^2 + \cdots + u_n(t_1)^2 \mathrm{d}t_n^2$$

on $\mathbb{T}^n$ with positive scalar curvature after induction by Fischer-Colbrie symmetrization. Now the contradiction can be derived from a direct computation.

In a more intuitive way the difference between the Fischer-Colbrie symmetrization and the conformal method can be summarized in the following way:

- In the conformal method, we use the stability inequality for a minimal hypersurface $\Sigma \subset M$ to conformally deform the induced metric on Σ to give it positive scalar curvature.
- When using the Fischer-Colbrie symmetrization process, the instrinsic geometry the stable minimal hypersurface $\Sigma \subset M$ is kept intact. Instead of using the stability inequality to build a deformation of Σ itself, we add an extra dimension $\mathbb{T}^1$ and use the stability inequality to build a suitable stretching for this extra factor such that $\Sigma \times \mathbb{T}^1$ has the same lower bound on the scalar curvature as M did and enjoy an extra symmetry.

The inductive use of Fischer-Colbrie–Schoen symmetrization is also called torical symmetrization by Gromov in [Gro18], where he gave an upper bound for the distance between the two boundary components of $\mathbb{T}^{n-1} \times [-1, 1]$ depending on a positive scalar curvature lower bound. In the following, we will show a homotopical 2-systole estimate for $\mathbb{S}^2 \times \mathbb{T}^n$ with positive scalar curvature using Fischer-Colbrie–Schoen symmetrization, which can be thought of as the 2D analog of Gromov's result.

4.2. **Small two spheres in p.s.c $\mathbb{S}^2 \times \mathbb{T}^n$.** In this subsection, we are going to give a proof for Theorem 1.2 based on the Fischer-Colbrie–Schoen symmetrization mentioned in previous subsection. We choose to present the proof only in dimension 4, this improves the readability while still showing the core idea of the proof. So let us focus on the following result.

Theorem 4.1. *Let (M, g) be a closed oriented Riemannian 4-manifold with scalar curvature R_g no less than 2, which admits a smooth map $f : M \to \mathbb{S}^2 \times \mathbb{T}^2$ with non-zero degree. Then M has non-trivial second homotopy group and it holds $\mathrm{sys}_2(M, g) \leq 4\pi$. Moreover, the equality holds if and only if the universal covering of (M, g) is isometric to the product manifold $(\mathbb{S}^2 \times \mathbb{R}^2, g_{round} + g_{euc})$, where g_{round} is the standard round metric on $\mathbb{S}^2$ and g_{euc} denotes the standard Euclidean metric on $\mathbb{R}^2$.*

First we show the desired homotopical 2-systole inequality.

Proof of the 2-systole bound. Let (θ_1, θ_2) be the standard coordinates on $\mathbb{T}^2$. Since $f : M \to \mathbb{S}^2 \times \mathbb{T}^2$ has non-zero degree, we see

$$f_* \left([M] \frown f^*(\mathrm{d}\theta_1)\right) = f_*([M]) \frown \mathrm{d}\theta_1 = (\deg f) \cdot [\mathbb{S}^2 \times \{*\} \times \mathbb{S}^1].$$

In particular, $[M] \frown f^*(\mathrm{d}\theta_1)$ is a non-trivial homology class in $H_3(M, \mathbf{Z})$. From geometric measure theory, there is a smooth embedded oriented minimal hypersurface Σ_1 with integer multiplicity homologous to $[M] \frown f^*(\mathrm{d}\theta_1)$, which has the least area in its homology class. As a result, Σ_1 is a two-sided stable minimal hypersurface in M. From the second variation formula, there is a smooth positive function u_1 on Σ_1 such that

$$-\Delta_1 u_1 + (\mathrm{Ric}_g(\nu_1, \nu_1) + |A_1|^2) u_1 = \lambda_1 u_1, \tag{4.1}$$

where Δ_1 is the Laplace operator of Σ_1 with the induced metric g_1', ν_1 is a chosen unit normal vector field on Σ_1, A_1 is the second fundamental form of Σ_1 with respect to ν_1, and λ_1 is a non-negative constant. Following Fischer-Colbrie-Schoen symmetrization, we define $M_1 = \Sigma_1 \times \mathbb{T}^1$ and

$$g_1 = g_1' + u_1^2 \mathrm{d}t_1^2.$$

It follows from a direct computation and the Gauss equation that

$$R_{g_1} = R_{g_1'} - \frac{2\Delta_1 u_1}{u_1} = R_g + |A_1|^2 + 2\lambda_1 \geq 2. \tag{4.2}$$

Denote

$$\pi_1 : \mathbb{S}^2 \times \mathbb{T}^2 \to \mathbb{S}^2 \times \mathbb{T}^1, \quad (p, \theta_1, \theta_2) \mapsto (p, \theta_2)$$

to be the projection map and $i_1 : \Sigma_1 \to \mathbb{S}^2 \times \mathbb{T}^2$ to be the inclusion map. Naturally, the composed map $f_1 = \pi_1 \circ f \circ i_1$ gives a smooth map from Σ_1 to $\mathbb{S}^2 \times \mathbb{S}^1$ with non-zero degree. As a consequence, M_1 admits a smooth map to $\mathbb{S}^2 \times \mathbb{T}^2$ with non-zero degree in the form of (f_1, id).

We are going to repeat above procedure. Notice that the homology class given by $[M_1] \frown f_1^*(\mathrm{d}\theta_2)$ is a non-trivial element in $H_3(M_1, \mathbf{Z})$. Again we can construct a smooth embedded oriented minimal hypersurface Σ_2 in M_1 with integer multiplicity homologous to $[M_1] \frown f_1^*(\mathrm{d}\theta_2)$ such that it has the least area in its homology class.

Since (M_1, g_1) has $\mathbb{T}^1$-invariance, it is reasonable to expect that Σ_2 splits as $\hat{\Sigma}_2 \times \mathbb{T}^1$. To see this, it suffices to show that the Killing field induced by the $\mathbf{T}^1$-isometry is tangent to Σ_2. Otherwise, the Killing field will induce a non-zero Jacobi function on Σ_2. The stability of Σ_2 then yields the positivity of the Jacobi function and the integral curve generated by the Killing field will have non-zero intersection number with Σ_2. From our choice of the homology class $[M_1] \frown f_1^*(\mathrm{d}\theta_2)$, the intersection should be zero and we obtain a contradiction. From the splitting of Σ_2 we obtain a smooth embedded surface $\hat{\Sigma}_2$ in Σ_1 with integer multiplicity. As a result, $\hat{\Sigma}_2$ is an embedded surface in M representing the homology class $[M] \frown (f^*\mathrm{d}\theta_1 \smile f^*\mathrm{d}\theta_2)$. Denote

$$\pi_2 : \mathbb{S}^2 \times \mathbb{T}^2 \to \mathbb{S}^2 \quad \text{and} \quad i_2 : \hat{\Sigma}_2 \to M.$$

Then the map $\pi_2 \circ f \circ i_2$ is a smooth map from $\hat{\Sigma}_2$ to $\mathbb{S}^2$ with non-zero degree.

Now we derive more useful information from the stability of Σ_2. As before, the stability inequality actually yields a positive function u_2 on Σ_2 such that

$$-\Delta_2 u_2 + (\mathrm{Ric}_{g_1}(\nu_1, \nu_1) + |A_2|^2) u_2 = \lambda_1 u_2, \quad \lambda_2 \geq 0. \tag{4.3}$$

Given the $\mathbb{T}^1$-invariance of M_1 and Σ_2, the Laplace operator Δ_2 and the curvature quantities above are all $\mathbb{T}^1$-invariant and so is the function u_2. In the following, we also view u_2 as a function on $\hat{\Sigma}_2$ denoted by $\hat{u}_2$. According to the Fischer-Colbrie symmetrization once again, we define

$$M_2 = \Sigma_2 \times \mathbb{T}^1 = \hat{\Sigma}_2 \times \mathbb{T}^2 \quad \text{and} \quad g_2 = g_2' + u_2^2 \mathrm{d}t_2^2 = \hat{g}_2 + \hat{u}_1^2 \mathrm{d}t_1^2 + \hat{u}_2^2 \mathrm{d}t_2^2, \quad \hat{u}_1 := u_1|_{\hat{\Sigma}_2},$$

where g_2' is the induced metric of Σ_2 from (M_1, g_1) and $\hat{g}_2$ is the induced metric of $\hat{\Sigma}_2$ in (M, g). A straightforward calculation shows

$$R_{g_2} = R_{g_2'} - \frac{\Delta_2 u_2}{u_2} = R_{g_1} + |A_2|^2 + 2\lambda_2 \geq 2 \tag{4.4}$$

and

$$R_{g_2} = R_{\hat{g}_2} - 2\left(\frac{\hat{\Delta}\hat{u}_1}{\hat{u}_1} + \frac{\hat{\Delta}\hat{u}_2}{\hat{u}_2}\right) - 2\hat{g}_2(\hat{\nabla}\log\hat{u}_1, \hat{\nabla}\log\hat{u}_2), \tag{4.5}$$

where $\hat{\Delta}$ and $\hat{\nabla}$ are Laplace and gradient operators of $(\hat{\Sigma}_2, \hat{g}_2)$. From (4.4) we know that $\hat{\Sigma}_2$ consists of spherical components. By integrating (4.5) on each component $\hat{\Sigma}_2'$ of $(\hat{\Sigma}_2, \hat{g}_2)$, we see

$$\begin{aligned} 2\,\mathrm{area}(\hat{\Sigma}_2', \hat{g}_2) &\leq \int_{\hat{\Sigma}_2'} R_{g_2}\,\mathrm{d}\sigma_{\hat{g}_2} \\ &= 4\pi\chi(\hat{\Sigma}_2') - \int_{\hat{\Sigma}_2'} |\hat{\nabla}\log\hat{u}_1|^2 + |\hat{\nabla}\log\hat{u}_1|^2\,\mathrm{d}\sigma_{\hat{g}_2} \\ &\qquad - \int_{\hat{\Sigma}_2'} |\hat{\nabla}(\log\hat{u}_1 + \log\hat{u}_2)|^2\,\mathrm{d}\sigma_{\hat{g}_2} \leq 8\pi. \end{aligned} \tag{4.6}$$

This yields that each component of $(\hat{\Sigma}_2, \hat{g}_2)$ has area no greater than 4π. Recall that the map $f_2 : \hat{\Sigma}_2 \to \mathbb{S}^2$ has non-zero degree, then at least one component of $\hat{\Sigma}_2$ is homotopical non-trivial in (M, g) and so we have $\mathrm{sys}_2(M, g) \leq 4\pi$. □

The rigidity will come from a more careful analysis. Before we continue the proof, let us keep in mind the following diagram that illustrates our construction above.

$$\begin{array}{ccccc} (\hat{\Sigma}_2, \hat{g}_2) & \xrightarrow{i} & (\Sigma_1, g_1') & \xrightarrow{i} & (M, g) \\ \big\downarrow{\scriptstyle \times\mathbb{T}^1} & & \big\downarrow{\scriptstyle \times\mathbb{T}^1} & & \\ (\Sigma_2, g_2') & \xrightarrow{i} & (M_1, g_1) & & \\ \big\downarrow{\scriptstyle \times\mathbb{T}^1} & & & & \\ (M_2, g_2) & & & & \end{array}$$

Proof of the equality case. Take a component $\hat{\Sigma}_2'$ of $\hat{\Sigma}_2$ that represents a non-trivial homology class. If we have $\mathrm{sys}_2(M, g) = 4\pi$, then $\mathrm{area}(\hat{\Sigma}_2', \hat{g}_2) = 4\pi$. From (4.6) it follows that $\hat{u}_1$ and $\hat{u}_2$ are positive constant functions on $\hat{\Sigma}_2'$ and then $(\hat{\Sigma}_2', \hat{g}_2)$ is isometric to the standard round sphere. Combined with (4.3) and (4.4), the component $\Sigma_2' = \hat{\Sigma}_2' \times \mathbb{S}^1$ of Σ is totally geodesic and the normal Ricci curvature vanishes identically on Σ_2'. Notice that the Jacobi operator of Σ_2' reduces to the Laplace operator. Now we borrow a nice idea from [BBN10] raised up by Bray, Brendle and Neves. It turns out that we can construct a foliation $\{\Sigma_{2,t}'\}_{-\epsilon<t<\epsilon}$ with $\Sigma_{2,0}'=\Sigma_2'$ consisting of hypersurfaces with constant mean curvature (abbreviated by CMC) from the implicit function theorem. If we can show that hypersurfaces $\{\Sigma_{2,t}'\}$ is area-minimizing in its homology class, then the same analysis as above yields that each $\Sigma_{2,t}'$ is totally geodesic and so (M_1,g_1) locally as the product manifold $\Sigma_2'\times(-\epsilon,\epsilon)$. This further implies a local isometry from $\Sigma_2'\times\mathbb{R}$ to (M_1,g_1) after an open and closed argument.

So let us focus on the area-minimizing property of $\Sigma_{2,t}'$. In fact, the underlying philosophy is that stable non-zero CMC hypersurfaces Σ possess more scalar curvature than stable minimal hypersurfaces since the stability now yields a positive function u satisfying

$$-\Delta u+(\mathrm{Ric}(\nu,\nu)+|A|^2)u=\lambda u,$$

where the term $|A|^2$ provides more positivity due to the non-zero mean curvature. Since $\mathrm{sys}_2(M,g)$ is not smaller than 4π, it forbids the existence of a stable non-zero CMC hypersurface Σ homologous to Σ_2' that splits as $\Sigma'\times\mathbb{T}^1$ due to our previous argument. This yields that the mean curvature of $\Sigma_{2,t}'$ with respect to ∂_t is non-positive when $t\geq 0$ and it is non-negative when $t\leq 0$. If this is not the case, say the mean curvature of $\Sigma_{2,t}'$ is a positive constant c for some positive t, then Σ_2' and $\Sigma_{2,t}'$ can serve as barriers and we can minimize the Brane functional $\mathcal{B}(\Omega)=\mathrm{area}(\partial\Omega)-\frac{c}{2}\,\mathrm{vol}(\Omega)$ to obtain a stable CMC hypersurface Σ homologous to Σ_2' splitting as $\Sigma'\times\mathbb{T}^1$, which leads to a contradiction. When the mean curvature of $\Sigma_{2,t}'$ behaves as described above, then the area of $\Sigma_{2,t}'$ is always no greater than Σ_2' and so it holds $\mathrm{area}(\Sigma_{2,t}')\equiv\mathrm{area}(\Sigma_2')$. As a result, $\Sigma_{2,t}'$ is area-minimizing in its homology class as Σ_2'.

Now, there is a local isometry $\Phi_2:\Sigma_2'\times\mathbb{R}\to M_1$. In particular, this yields that the function u_1 is a positive constant on Σ_1 and there is a local isometry $\phi_2:\hat{\Sigma}_2'\times\mathbb{R}\to\Sigma_1$. Also, we see that the scalar curvature R_{g_1} is identical to 2. Combined with (4.1) and (4.2), this implies that Σ_1 is totally geodesic in (M,g) and the normal Ricci curvature vanishes along Σ_1. Repeating above analysis, we obtain a local isometry $\Phi_1:\Sigma_1\times\mathbb{R}\to M$. The composition of ϕ_2 and Φ_1 gives a local isometry from $\mathbb{S}^2\times\mathbb{R}^2$ to (M,g), which yields the universal covering of (M,g) is the standard product manifold $\mathbb{S}^2\times\mathbb{R}^2$. □

We emphasize that the above argument is still valid for higher dimensions no greater than seven without any change.

5. Further quantitative implications using μ-bubbles

In previous sections, we have seen various applications of minimal hypersurface in problems involving scalar curvature. However, one should not limit his sight only to minimal hypersurfaces. In fact, there is a broader class of hypersurfaces called *μ-bubbles*, whose variation theory is also powerful. They were introduced by Gromov in his IHES lectures [Gro19].

Let (M,g) be a Riemannian manifold and μ be a Borel measure on M. A hypersurface Σ bounding a region Ω is called a μ-bubble if Ω is a critical point of the functional

$$\mathcal{B}(\Omega)=\mathrm{area}(\partial\Omega)-\mu(\Omega).$$

Usually, we take the Borel measure μ to be the one associated with a smooth function h on M. Namely,

$$\mu(\Omega) = \int_\Omega h \, \mathrm{d}\,\mathrm{vol}_g \,.$$

In this case, a μ-bubble is just an h-boundary (i.e. a boundary with mean curvature h with respect to the unit outer normal).

The extra freedom given by the choice of μ can be used to ensure the existence of stable μ-bubbles in situations where a stable minimal hypersurface would be hard to find or control.

In the following, we discuss several applications of μ-bubbles to the estimation of the homotopical 2-systole of manifolds with positive scalar curvature.

5.1. **The width of p.s.c metrics on** $[-1,1]\times\mathbb{T}^{n-1}$**.** Assume for a minute we don't know that $\mathbb{T}^n$ doesn't admit a metric with positive scalar curvature and try to build a metric with scalar curvature $n(n-1)$ on $\mathbb{T}^n$ in the most naive way: consider a warped product $g = dt^2 + f(t)^2 dx^2$ on $(-\varepsilon,\varepsilon)\times\mathbb{T}^{n-1}$, where dx^2 is a flat metric on $\mathbb{T}^{n-1}$. A routine computation shows that

$$R_g = -2(n-1)\left(\frac{f'(t)}{f(t)}\right)' - n(n-1)\left(\frac{f'(t)}{f(t)}\right)^2.$$

By requiring the scalar curvature to be $n(n-1)$ we get an ODE which can be solved, and get that $f(t) = \left(\cos\frac{nt}{2}\right)^{\frac{2}{n}}$ and thus that g is at best defined on $(-\frac{\pi}{n},\frac{\pi}{n})$. In particular the two boundary components cannot be further than $\frac{2\pi}{n}$ apart.

In [Gro18], Gromov proved that this upper bound holds in a much more general setting:

Theorem 5.1. *Let $n\leq 7$ and g be a metric on $[-1,1]\times\mathbb{T}^{n-1}$ with $R_g\geq n(n-1)$ then:*

$$d_g\left(\{-1\}\times\mathbb{T}^{n-1},\{1\}\times\mathbb{T}^{n-1}\right)\leq\frac{2\pi}{n}.$$

This is another kind of quantitative result involving scalar curvature, an interesting fact is that the comparison is not made with a constant sectionnal curvature metric but with the constant scalar curvature metric on $(-\frac{\pi}{n},\frac{\pi}{n})\times\mathbb{T}^{n-1}$ we built before.

This results also gives another proof of the impossibility to endow $\mathbb{T}^n$ with a positive scalar curvature: such a metric would lift to a complete metric with uniformly positive scalar curvature using the covering $\mathbb{R}\times\mathbb{T}^{n-1}\to\mathbb{T}^n$, and the induced metric on $[-L,L]\times\mathbb{T}^{n-1}$ would boundary components as far appart as wanted provided L is big enough.

Sketch of proof using FCS symmetrization. The proof given in [Gro18] is an application of Fischer-Colbrie–Schoen symmetrization with a twist: one needs to consider hypersurfaces with boundary. To keep this survey short, we will not address the technical issues raised by the presence of boundaries.

One starts by minimizing the area among all hypersurfaces which are homologous to $[-1,1]\times\mathbb{T}^{n-2}\times\{*\}$ relative to the boundary of $[-1,1]\times\mathbb{T}^{n-1}$. Let us call Σ^{n-1} such a minimzer. The Fischer-Colbrie–Schoen symmetrization process allows us to build a metric $g_n = g_\Sigma + (u_n d\theta_n)^2$ on $\Sigma^{n-1}\times\mathbb{T}^1$ with:

- $u_n:\Sigma\to\mathbb{R}$ a positive function.
- g_Σ being the metric induced by g on Σ^{n-1}.

- $g_n = g_\Sigma + (u_n(x)d\theta_n)^2$ has scalar curvature at least $n(n-1)$.

Let us notice that since Σ^{n-1} is isometrically embedded in M in a boundary compatible way, the distance between the boundary component of Σ^{n-1} is bigger than the distance between the boundary components of M. And since the metric g_n on $\Sigma^{n-1}\times\mathbb{T}^1$ is invariant by the action of $\mathbb{R}$ on $\mathbb{T}^1$, the distance between the boundary components in $\Sigma^{n-1}\times\mathbb{T}^1$ is the same as it is Σ^{n-1}.

Assume for a while that Σ^{n-1} is diffeomorphic to $[-1,1]\times\mathbb{T}^{n-2}\times\{*\}$. We could then minimize area among hypersurfaces of the form $S^{n-2}\times\mathbb{T}^1\subset\Sigma^{n-1}\times\mathbb{T}^1$ where $S^{n-2}\subset\Sigma^{n-1}$ is homologous to $[-1,1]\times\mathbb{T}^{n-3}\times\{*\}\subset[-1,1]\times\mathbb{T}^{n-2}$ relative to the boundary, let us call $\tilde\Sigma^{n-2}$ a minimizer. An application of the Fischer-Colbrie–Schoen symmetrization then gives a metric $g_{\tilde\Sigma}+u_{n-1}^2d\theta_{n-1}^2+u_n^2d\theta_n^2$ on $\tilde\Sigma^{n-2}\times\mathbb{T}^2$ with scalar curvature greater than $n(n-1)$. Iterating this construction we gain more and more symmetry and in the end get a metric $dt^2+\sum_{i=2}^n(u_i(t)d\theta_i)^2$ on $I\times\mathbb{T}^{n-1}$ with scalar curvature bigger $n(n-1)$ where I is an interval. This lower bound on the scalar curvature gives a differential inequality on the functions u_i which in turns bounds the length of I by $\frac{2\pi}{n}$.

The fact that Σ^{n-1} may be topologically more complicated than $[-1,1]\times\mathbb{T}^{n-2}\times\{*\}$ is handled by the use of non zero degree maps to $[-1,1]\times\mathbb{T}^{n-1}$ at each step. □

5.2. **A simpler proof using μ-bubbles.** Looking at the model $(-\pi/n,\pi/n)\times\mathbb{T}^{n-1}$ with the metric $dt^2+\left(\cos\frac{nt}{2}\right)^{\frac{4}{n}}dx^2$, the previous proof used the fact that $(-\pi/n,\pi/n)\times\mathbb{T}^{n-2}\times\{*\}$ is a stable minimal hypersurface.

If we want to find a proof that avoids dealing with hypersurfaces with boundary, we would need to find a stable closed minimal hypersurface in the model. The hypersurface $\Sigma_0=\{0\}\times\mathbb{T}^{n-1}$ is minimal but not stable. In his IHES lectures [Gro19], Gromov gave a new proof of the previous theorem by using μ-bubbles with μ chosed in such a way that in the model the hypersurface Σ_0 is a stable μ-bubble.

First let us focus on the existence of μ-bubbles. For convenience, we introduce the following definition.

Definition 5.2. A triple $(M,\partial_\pm,g)$ is called a Riemannian band if (M,g) is a compact Riemannian manifold with non-empty boundary, where ∂_+ and ∂_- are two portions of the boundary $\partial\Omega$ such that

(i) ∂_+ and ∂_- are disjoint and $\partial\Omega=\partial_+\cup\partial_-$;
(ii) ∂_+ and ∂_- are union of boundary components of Ω.

Let $(M^n,g,\partial_\pm)$ be a Riemannian band and Σ_0 be a fixed closed hypersurface separating ∂_- and ∂_+. We denote Ω_0 to be the region associated to Σ_0 such that $\partial\Omega_0=\partial_-\cup\Sigma_0$. Let

$$\mathcal{C}=\{\text{Caccippoli sets }\Omega\text{ in }M\text{ such that }\Omega\Delta\Omega_0\Subset\mathring{M}\},$$

where $\mathring{M}$ denotes the interior of M. Given a smooth function h defined over $\mathring{M}$, we consider

$$\mathcal{A}^h(\Omega)=\mathcal{H}^{n-1}(\partial\Omega\cap\mathring{M})-\int_{\mathring{M}}(\chi_\Omega-\chi_{\Omega_0})h\,\mathrm{d}\mathcal{H}^n_g.$$

The use of $\mathcal{A}^h$ instead of the functional $\mathcal{B}$ defined above is necessary to handle noncompact ambient space where h may not be integrable on some Ω that we would like include. $\mathcal{A}^h$ and $\mathcal{B}$ differ by a constant if they are both defined, hence the variation formulas will not be affected by this change.

An advantage of μ-bubbles is that they always exist on a given Riemannian band for a suitably imposed function h.

Proposition 5.3. *For $n \le 7$, if*

$$\lim_{x\to\partial_-} h(x) = +\infty \quad \text{and} \quad \lim_{x\to\partial_+} h(x) = -\infty, \tag{5.1}$$

then there exists a smooth minimizer $\hat{\Omega} \in \mathcal{C}$ for $\mathcal{A}^h$.

Proof. Denote

$$I = \inf\{\mathcal{A}^h(\Omega) : \Omega \in \mathcal{C}\}.$$

First we show $I > -\infty$. For any $s > 0$, we denote

$$\Sigma_s^\pm = \{x \in \mathring{M} : \operatorname{dist}(x, \partial_\pm) = s\}.$$

Clearly, $\Sigma_s^\pm$ is a foliation around $\partial_\pm$ when s is small. From (5.1) we can assume $H_s^- \le h\circ\phi$ and $H_s^+ \le -h\circ\phi$ for $s \le s_0$, where s_0 is a small positive constant and $H_s^\pm$ is the mean curvature of $\Sigma_s^\pm$ with respect to ∂_s. Let $\Omega_s^\pm$ be the region enclosed by $\Sigma_s^\pm$ and $\partial_\pm$. Possibly decreasing the value of s_0, we can construct a smooth vector field X such that $X = \partial_s$ on $\Omega_{s_0}^\pm$. It is clear that

$$\operatorname{div}_g X = H_s^- \le h \quad \text{in} \quad \Omega_{s_0}^-$$

and

$$\operatorname{div}_g X = H_s^+ \le -h \quad \text{in} \quad \Omega_{s_0}^+.$$

Notice that for any region $\Omega \in \mathcal{C}$ we have the following estimate

$$\begin{aligned}
&\mathcal{A}^h(\Omega \cup \Omega_{s_0}^- \backslash \Omega_{s_0}^+) - \mathcal{A}^h(\Omega)\\
=&\mathcal{H}^{n-1}(\partial\Omega_{s_0}^- \backslash \Omega) - \mathcal{H}^{n-1}(\partial^*\Omega \cap \Omega_{s_0}^-) + \mathcal{H}^{n-1}(\partial\Omega_{s_0}^+ \cap \Omega)\\
&\quad - \mathcal{H}^{n-1}(\partial^*\Omega \cap \Omega_{s_0}^+) - \int_{\Omega_{s_0}^- \backslash \Omega} h \,\mathrm{d}\mathcal{H}_g^n + \int_{\Omega\cap\Omega_{s_0}^+} h \,\mathrm{d}\mathcal{H}_g^n\\
\le&\mathcal{H}^{n-1}(\partial\Omega_{s_0}^- \backslash \Omega) - \mathcal{H}^{n-1}(\partial^*\Omega \cap \Omega_{s_0}^-) + \mathcal{H}^{n-1}(\partial\Omega_{s_0}^+ \cap \Omega)\\
&\quad - \mathcal{H}^{n-1}(\partial^*\Omega \cap \Omega_{s_0}^+) - \int_{\Omega_{s_0}^- \backslash \Omega} \operatorname{div}_g X \,\mathrm{d}\mathcal{H}_g^n - \int_{\Omega\cap\Omega_{s_0}^+} \operatorname{div}_g X \,\mathrm{d}\mathcal{H}_g^n\\
\le&0,
\end{aligned}$$

since

$$\begin{aligned}
\int_{\Omega_{s_0}^- \backslash \Omega} \operatorname{div}_g X \,\mathrm{d}\mathcal{H}_g^n &= \int_{\partial^*(\Omega_{s_0}^- \backslash \Omega)} \langle X, \nu\rangle_g \,\mathrm{d}\mathcal{H}_g^{n-1}\\
&\ge \mathcal{H}^{n-1}(\partial\Omega_{s_0}^- \backslash \Omega) - \mathcal{H}^{n-1}(\partial^*\Omega \cap \Omega_{s_0}^-)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega\cap\Omega_{s_0}^+} \operatorname{div}_g X \,\mathrm{d}\mathcal{H}_g^n &= \int_{\partial^*(\Omega\cap\Omega_{s_0}^+)} \langle X, \nu\rangle_g \,\mathrm{d}\mathcal{H}_g^{n-1}\\
&\ge \mathcal{H}^{n-1}(\partial\Omega_{s_0}^+ \cap \Omega) - \mathcal{H}^{n-1}(\partial^*\Omega \cap \Omega_{s_0}^+).
\end{aligned}$$

It follows

$$\mathcal{A}^h(\Omega) \ge \mathcal{A}^h(\Omega \cup \Omega_{s_0}^- \backslash \Omega_{s_0}^+) \ge -C\mathcal{H}^n(M, g), \quad \forall\, \Omega \in \mathcal{C},$$

where C is a universal constant such that $|h| \le C$ on $M - \Omega_{s_0}^- \cup \Omega_{s_0}^+$. Hence $I > -\infty$.

Now we establish the existence of a smooth minimizer for $\mathcal{A}^h$ in $\mathcal{C}$. Let Ω_k be a sequence of regions in $\mathcal{C}$ such that $\mathcal{A}^h(\Omega_k) \to I$ as $k \to \infty$. According to the discussion above we can assume $\Omega_k \Delta \Omega_0 \subset M - \Omega^-_{s_0} \cup \Omega^+_{s_0}$. For k large enough there holds

$$\mathcal{H}^{n-1}(\partial^* \Omega_k) \leq I + 1 + C\mathcal{H}^n(M, g).$$

From the compactness of Caccippoli sets, after taking the limit of Ω_k we can obtain $\hat{\Omega} \in \mathcal{C}$ with $\mathcal{A}^h(\hat{\Omega}) = I$. The smoothness of $\partial\hat{\Omega}$ comes from the regularity theorem [ZZ20, Theorem 2.2]. □

Proof of Theorem 5.1 using μ-bubbles. Let $\mathcal{H}(t)$ be the mean curvature of the hypersurface $\{t\} \times \mathbb{T}^{n-1}$ in the model $(-\pi/n, \pi/n) \times \mathbb{T}^{n-1}$, we will use this mean curvature in the model to build a suitable μ-bubble functional. Note that $2\mathcal{H}' = -n(n-1) - \frac{n}{n-1}\mathcal{H}^2$.

The proof goes by contradiction. Assume $([-1,1] \times \mathbb{T}^{n-1}, g)$ has scalar curvature at least $n(n-1)$ and that $d_g\left(\{-1\} \times \mathbb{T}^{n-1}, \{1\} \times \mathbb{T}^{n-1}\right) > \frac{2\pi}{n}$. This allows us to build a smooth surjective 1-Lipschitz function $\phi : [-1,1] \times \mathbb{T}^{n-1} \to (-\frac{\pi+\varepsilon}{n}; \frac{\pi+\varepsilon}{n})$ whose level sets separates the two components of the boundary of $\phi : [-1,1] \times \mathbb{T}^{n-1}$.

Set:

$$h : [-1,1] \times \mathbb{T}^{n-1} \to [-\infty, +\infty]$$

$$x \mapsto \begin{cases} +\infty & \text{if } \phi(x) \leq -\frac{\pi}{n} \\ \mathcal{H}(\phi(x)) & \text{if } \phi(x) \in (-\pi/n, \pi/n) \\ -\infty & \text{if } \phi(x) \geq \frac{\pi}{n} \end{cases}$$

The behavior of h close to the boundary of $[-1,1] \times \mathbb{T}^{n-1}$ ensures that a minimizer Ω of the $\mathcal{A}^h$ functional defined on $h^{-1}(]-\infty, +\infty[)$ with this choice of h and a suitable choice of Ω_0 exists and stays where h is finite, see Proposition 5.3 above. The positivity of the second variation of $\mathcal{B}$ gives that for any function $f : \partial\Omega \to \mathbb{R}$:

$$\int_\Sigma |\nabla f|^2 - \tfrac{1}{2}\left(R_M - R_\Sigma + |\,\mathrm{A}\,|^2 + 2\langle \nabla h, \nu \rangle + h^2\right) f^2 d\sigma \geq 0$$

where $\Sigma = \partial\Omega$ and A and ν are the second fundamental form and unit normal to Σ.

Using that on Σ we have $h^2 = H_\Sigma^2 \leq (n-2)|\,\mathrm{A}\,|^2$, this implies that the operator:

$$-\Delta_\Sigma + \tfrac{1}{2}R_\Sigma - \tfrac{1}{2}\left(R_M - 2|\nabla h| + \tfrac{n}{n-1}h^2\right)$$

is nonnegative on $C^\infty(\Sigma)$. Moreover, our choice of h together with the fact that $R_M \geq n(n-1)$ will ensure that the term in between the parenthesis above is nonnegative. The will then show that $-\Delta_\Sigma + \frac{1}{2}R_\Sigma \geq 0$, which implies as in section 3 allow us to show the conformal Laplacian of $\Sigma = \partial\Omega$ is positive, hence Σ admits a metric with positive scalar curvature. Since the map $\Sigma \subset [-1,1] \times \mathbb{T}^{n-1} \to \mathbb{T}^{n-1}$ has non zero degree, this is a contradiction. □

Using the same proof without the topological assumption and using Fischer-Colbrie–Schoen symmetrization instead of the conformal method in the last step, Gromov also proved:

Theorem 5.4. *Let $(M^n, \partial_\pm, g)$ $(n \leq 7)$ be a Riemannian band. Assume that*

$$R_g \geq \frac{4(n-1)\pi^2}{n\, d_g(\partial_-, \partial_+)^2} + \delta$$

for some $\delta > 0$. Then there exists

- *an hypersurface Σ which separates ∂_- and ∂_+,*
- *a positive function $u : \Sigma \to \mathbb{R}$,*

such that the metric $h = g|_\Sigma + u^2 dt^2$ on $\Sigma \times \mathbb{R}$ has $R_h \geq \delta$.

The conclusion of this theorem can be summarized in the following way: if (M, g) has two boundary components ∂_- and ∂_+ which are $\frac{2\pi}{n}$ appart and has scalar curvature uniformly bigger than $n(n-1)$ by an amout of δ, then this excess δ of scalar curvature compared to the $[-1, 1] \times \mathbb{T}^{n-1}$ situation can be used to build a warped product metric of scalar curvature greater than δ on $\Sigma \times \mathbb{S}^1$ where Σ is an hypersurface in M separating ∂_- and ∂_+.

5.3. **Small two-spheres in p.s.c $\mathbb{S}^2 \times \mathbb{S}^2$.** As an application of this idea, we give an upper bound for the 2-systole of some metrics with positive scalar curvature on $\mathbb{S}^2 \times \mathbb{S}^2$, proved by the first author in [Ric20]. Let us first recall that no universal upper bound is known for the 2-systole of $(\mathbb{S}^2 \times \mathbb{S}^2, g)$ with $R_g \geq 4$.

In order to state the result, we need some terminology. Let $\mathcal{S}_\ell$ be the set of embedded surfaces $S \subset \mathbb{S}^2 \times \mathbb{S}^2$ which are homologous to $\mathbb{S}^2 \times \{*\}$. Let $\mathrm{sys}_\ell(g)$ be the infimum of the area of 2-spheres in $\mathcal{S}_\ell$. By definition, $\mathrm{sys}_2(g) \leq \mathrm{sys}_\ell(g)$.

We also introduce a coarse measure of the size $\mathcal{S}_\ell$, called the left stretch of g and defined by $\mathrm{str}_\ell(g) = \sup_{S_1, S_2 \in \mathcal{S}_\ell} d_g(S_1, S_2)$. We can now state the estimate:

Theorem 5.5. *Let g be a metric on $\mathbb{S}^2 \times \mathbb{S}^2$ with $R_g \geq 4$.*

If $s = \mathrm{str}_\ell(\mathbb{S}^2 \times \mathbb{S}^2, g) > \frac{\sqrt{3}\pi}{2}$, then $\mathrm{sys}_\ell(\mathbb{S}^2 \times \mathbb{S}^2, g) \leq \frac{8\pi s^2}{4s^2 - 3\pi^2}$. Moreover there is an embedded 2-sphere whose area is at most $\frac{8\pi s^2}{4s^2 - 3\pi^2}$.

Sketch of proof. We will show here in a qualitative way why a large enough lower bound on str_ℓ gives an upper bound on sys_ℓ, the precise inequality is obtained by writing the precise inequality at each step.

Let us assume that $s = \mathrm{str}_\ell$ is big. Then we can find two surfaces S_1 and S_2 in $\mathcal{S}_\ell$ which are far away from each other. Thus $\tilde{M} = M \backslash (S_1 \cup S_2)$ has two boundary components which are about str_ℓ apart.

Since g has $R_g \geq 4$, we can apply Theorem 5.4 provided $s > \frac{\sqrt{3}\pi}{2}$. This implies the existence of an hypersurface Σ which separates S_1 and S_2 together with a function $u : \Sigma \to \mathbb{R}$ such that the metric $g_\Sigma + (ud\theta)^2$ on $\Sigma \times \mathbb{S}^1$ has strictly positive scalar curvature. Moreover a topological argument shows that there is a non trivial map $\Sigma \to \mathbb{S}^2 \times \mathbb{T}^1$, and thus a non trivial map $\Sigma \times \mathbb{T}^1 \to \mathbb{S}^2 \times \mathbb{T}^2$.

Then we use Theorem 1.2 to conclude that there is a small 2-sphere in $\Sigma \times \mathbb{T}^1$ which (thanks to the $\mathbb{R}$-symmetry of $g_\Sigma + (ud\theta)^2$) is indeed a small 2-sphere in Σ. Since Σ is isometrically embedded in $\mathbb{S}^2 \times \mathbb{S}^2$, this gives us a small 2-sphere in $\mathbb{S}^2 \times \mathbb{S}^2$, which can be shown to belong to $\mathcal{S}_\ell$. □

5.4. **Rigidity results using μ-bubbles.** In this subsection, we deal with Theorem 1.4 and Theorem 1.5. As we will see later, μ-bubbles have very nice compactness property to guarantee the existence of an area-minimizing hypersurface in non-compact complete manifolds sometimes.

In our later application, we will use the following family of functions.

Lemma 5.6. *For any $\epsilon \in (0, 1)$, there is a function*

$$h_\epsilon : \left(-\frac{1}{n\epsilon}, \frac{1}{n\epsilon}\right) \to (-\infty, +\infty)$$

such that

(1) h_ϵ *satisfies*

$$\frac{n}{n-1}h_\epsilon^2 + 2h_\epsilon' = n(n-1)\epsilon^2 \quad on \quad \left(-\frac{1}{n\epsilon}, -\frac{1}{2n}\right] \cup \left[\frac{1}{2n}, \frac{1}{n\epsilon}\right)$$

and there is a universal constant $C = C(n)$ *so that*

$$\sup_{-\frac{1}{2n}\le t\le \frac{1}{2n}} \left|\frac{n}{n-1}h_\epsilon^2 + 2h_\epsilon'\right| \le C\epsilon.$$

(2) $h_\epsilon' < 0$ *and*

$$\lim_{t\to\mp\frac{1}{n\epsilon}} h_\epsilon(t) = \pm\infty.$$

(3) *As* $\epsilon \to 0$, h_ϵ *converge smoothly to* 0 *on any closed interval.*

Proof. Let

$$h_\epsilon^+ : \left(-\frac{1}{n\epsilon}, +\infty\right), \quad t \mapsto (n-1)\epsilon \coth\left(\frac{n\epsilon t + 1}{2}\right)$$

and

$$h_\epsilon^- : \left(-\infty, \frac{1}{n\epsilon}\right), \quad t \mapsto -(n-1)\epsilon \coth\left(\frac{-n\epsilon t + 1}{2}\right).$$

Through a straightforward calculation we see that h_ϵ^+ and h_ϵ^- are solutions to the equation

$$\frac{n}{n-1}h^2 + 2h' = n(n-1)\epsilon^2.$$

We now glue h_ϵ^+ and h_ϵ^- to obtain the desired function h_ϵ. Fix a nonnegative smooth function $\bar\eta$ with compact support contained in $(-\frac{1}{2n}, \frac{1}{2n})$ and define

$$\eta(t) = \left(\int_{-\infty}^{+\infty} \bar\eta(s)\,\mathrm{d}s\right)^{-1} \int_{-\infty}^{t} \bar\eta(s)\,\mathrm{d}s.$$

Clearly, η is smooth with $\eta' \ge 0$ and $0 \le \eta \le 1$. Furthermore, it satisfies $\eta \equiv 0$ on $(-\infty, -\frac{1}{2n}]$ and $\eta \equiv 1$ on $[\frac{1}{2n}, +\infty)$. Denote

$$h_\epsilon = (1-\eta)h_\epsilon^+ + \eta h_\epsilon^-.$$

The proof is now completed by verifying listed properties one by one. □

Now we show how to prove Theorem 1.4 and Theorem 1.5 with μ-bubbles. The key idea is to approximate an area-minimizing hypersurface by μ-bubbles.

Theorem 5.7. *Let* (M, g) *be a complete open Riemannian manifold with non-trivial second homotopy group. If the scalar curvature of* (M, g) *is no less than* 2, *then it holds* $\mathrm{sys}_2(M, g) \le 4\pi$, *where the equality holds if and only if the universal covering of* (M, g) *is isometric to the product manifold* $(\mathbb{S}^2 \times \mathbb{R}, g_{round} + \mathrm{d}t^2)$.

Proof. Without loss of generality, we can assume M to be simply connected. In this case, we can find an embedded 2-sphere Σ_0 in M from the sphere theorem. Clearly Σ_0 is also homologically non-trivial due to the Hurewicz theorem. So Σ_0 divides M into two unbounded connected component and there is a proper smooth function $\phi : M \to (-\infty, +\infty)$ with $|\mathrm{d}\phi|_g < 1$ and $\phi^{-1}(0) = \Sigma_0$. Denote $\Omega_0 = \{\phi < 0\}$. From Sard's theorem, there is a sequence of $\epsilon_k \to 0$ such that

$$M_k = \phi^{-1}\left(\left[-\frac{1}{3\epsilon_k}, \frac{1}{3\epsilon_k}\right]\right)$$

is a Riemannian band with

$$\partial_\pm = \phi^{-1}\left(\pm\frac{1}{3\epsilon_k}\right).$$

Consider the functional

$$\mathcal{A}_k(\Omega) = \mathcal{H}^2_g(\partial^*\Omega) - \int_M (\chi_\Omega - \chi_{\Omega_0}) h_{\epsilon_k}\circ\phi \,\mathrm{d}\mathcal{H}^3_g$$

for all Caccipoli sets Ω such that $\Omega\Delta\Omega_0 \Subset \mathring{M}_k$. It follows from Proposition 5.3 that there is a smooth minimizer Ω_k with $\Omega\Delta\Omega_0 \Subset \mathring{M}_k$ for functional $\mathcal{A}_k$. For each component Σ_k of $\partial\Omega_k$, the second variation formula for $\mathcal{A}_k$ yields

$$\int_{\Sigma_k} |\nabla_k\psi|^2 - (\mathrm{Ric}_g(\nu_k,\nu_k) + |A_k|^2 - \nu_k(h_{\epsilon_k}\circ\phi))\psi^2 \,\mathrm{d}\sigma_k \geq 0,$$

where ∇_k and $\mathrm{d}\sigma_k$ is the gradient operator and area element of Σ_k with the induced metric, ν_k is the unit outer normal of Σ_k with respect to Ω_k and A_k is the corresponding second fundamental form. After taking the test function $\psi\equiv 1$ and applying the Schoen-Yau's trick, we see

$$\int_{\Sigma_k} R_g + |\mathring{A}_k|^2 + \left(\frac{3}{2}h^2_{\epsilon_k} + 2h'_{\epsilon_k}\right)\circ\phi\,\mathrm{d}\sigma_k \leq 4\pi\chi(\Sigma_k) \leq 8\pi. \tag{5.2}$$

Due to the facts $R_g \geq 2$ and

$$\frac{3}{2}h^2_{\epsilon_k} + 2h'_{\epsilon_k} \geq -C\epsilon_k \to 0,$$

we conclude that Σ_k is a sphere with area no greater than $8\pi(2-C\epsilon_k)^{-1}$ for k large enough. Since $\partial\Omega_k$ is homologous to Σ_0, it represents a non-trivial homology class in $H_2(M,\mathbf{Z})$. In particular, at least one component of $\partial\Omega_k$ is homotopically non-trivial, which yields $\pi_2(M)\neq 0$ and

$$\mathrm{sys}_2(M,g) \leq 8\pi(2-C\epsilon_k)^{-1} \to 4\pi, \quad \text{as} \quad k\to\infty.$$

Now we show the desired rigidity result under the assumption $\mathrm{sys}_2(M,g) = 4\pi$. From previous discussion we can pick up a connected component Σ_k of $\partial\Omega_k$ representing a non-trivial class. Our assumption forces Σ_k to have area no less than 4π and so (5.2) yields non-empty intersection of Σ_k and the fixed compact set

$$K = \phi^{-1}\left(\left[-\frac{1}{6},\frac{1}{6}\right]\right).$$

From direct comparison we have $\mathrm{area}(\Sigma_k)\leq \mathrm{area}(\Sigma_0)$. From the curvature estimate (see [ZZ20, Theorem 3.6]) and the minimizing property, Σ_k converges smoothly to an area-minimizing boundary Σ with multiplicity one, which has area no greater than 4π and non-empty intersection with K. It follows from [GL83, Theorem 8.8] that Σ is a sphere. Now surfaces Σ_k turn out to graphs over Σ and so Σ also represents a non-trivial homology class. From $\mathrm{sys}_2(M,g)=4\pi$ we know that Σ has the least area in its homology class. The rigidity now comes from the argument in [BBN10]. □

The idea to prove Theorem 1.5 is similar to the above but we need more effort to show the compactness of the limit of Σ_k. As before, we only work in dimension 4 since no essential difference will occur in higher dimensions no greater than seven.

Theorem 5.8. *Let (M,g) be a complete, oriented, open Riemannian 4-manifold with scalar curvature R_g no less than 2, which admits a smooth proper map $f : M \to \mathbb{S}^2\times\mathbb{S}^1\times\mathbb{R}$. Then M has non-trivial second homotopy group and it holds $\mathrm{sys}_2(M,g)\leq 4\pi$, where the equality holds if and only if the universal covering of (M,g) is isometric to the product manifold $(\mathbb{S}^2\times\mathbb{R}^2, g_{round}+g_{euc})$.*

Proof. For convenience, we write

$$f=(f_1,f_2): M\to(\mathbb{S}^2\times\mathbb{S}^1)\times\mathbb{R}.$$

From Sard's theorem we can assume that $\Sigma_0=f_2^{-1}(0)$ is an embedded hypersurface in M. It is not difficult to construct a smooth function $\phi: M\to(-\infty,+\infty)$ with $|\mathrm{d}\phi|_g<1$ and $\phi^{-1}(0)=\Sigma_0$. Denote $\Omega_0=\{\phi<0\}$. Again there is a sequence of $\epsilon_k\to 0$ such that

$$M_k=\phi^{-1}\left(\left[-\frac{1}{4\epsilon_k},\frac{1}{4\epsilon_k}\right]\right)$$

is a Riemannian band with

$$\partial_\pm=\phi^{-1}\left(\pm\frac{1}{4\epsilon_k}\right).$$

As before, we consider the functional

$$\mathcal{A}_k(\Omega)=\mathcal{H}^3_g(\partial^*\Omega)-\int_M(\chi_\Omega-\chi_{\Omega_0})h_{\epsilon_k}\circ\phi\,\mathrm{d}\mathcal{H}^4_g$$

for all Caccipoli sets Ω such that $\Omega\Delta\Omega_0\Subset\mathring{M}_k$. From Proposition 5.3 there is a smooth minimizer Ω_k with $\Omega\Delta\Omega_0\Subset\mathring{M}_k$ for functional $\mathcal{A}_k$. Since $\partial\Omega_k$ is homologous to Σ_0, the restriction of f_1 to $\partial\Omega_k$ gives a smooth map to $\mathbb{S}^2\times\mathbb{S}^1$ with non-zero degree and the same thing holds for some component Σ_k of $\partial\Omega_k$. The stability of Σ_k yields a positive smooth function u_k and a non-negative constant λ_k such that

$$-\Delta_k u_k-(\mathrm{Ric}_g(\nu_k,\nu_k)+|A_k|^2-\nu_k(h_{\epsilon_k}\circ\phi))u_k=\lambda_k u_k. \tag{5.3}$$

Denote g_k to be the induced metric of Σ_k from (M,g). Following Fischer-Colbrie-Schoen symmetrization, we can construct a new manifold $(\tilde\Sigma_k,\tilde g_k)$ with $\tilde\Sigma_k=\Sigma_k\times\mathbb{S}^1$ and $\tilde g_k=g_k+u_k^2\mathrm{d}t^2$. From a direct calculation and the Gauss equation, we have

$$\begin{aligned}R_{\tilde g_k}&=R_{g_k}-\frac{2\Delta_k u_k}{u_k}\\&\geq R_g+\left(\frac{4}{3}h^2_{\epsilon_k}+2h'_{\epsilon_k}\right)\circ\phi\\&\geq 2-C\epsilon_k.\end{aligned}$$

Notice that the map $F_k=(f_1|_{\Sigma_k},\mathrm{id})$ is a smooth map from $\tilde\Sigma_k$ to $\mathbb{S}^2\times\mathbb{T}^2$ with non-zero degree. It follows from Theorem 1.2 that Σ_k has non-trivial second homotopy group and

$$\mathrm{sys}_2(\tilde\Sigma_k,\tilde g_k)\leq 8\pi\,(2-C\epsilon_k)^{-1}.$$

Moreover, it follows from the proof of Theorem 1.2 that there is an embedded 2-sphere S_k in Σ_k with area no greater than $8\pi\,(2-C\epsilon_k)^{-1}$ and the map $\pi\circ f_1|_{S_k}: S_k\to\mathbb{S}^2$ has non-zero degree, where $\pi:\mathbb{S}^2\times\mathbb{S}^1\to\mathbb{S}^2$ denotes the projection map. In particular, S_k represents a non-trivial homology class in $H_2(M,\mathbf{Z})$ and so

$$\mathrm{sys}_2(M,g)\leq 8\pi\,(2-C\epsilon_k)^{-1}\to 4\pi,\quad\text{as}\quad k\to\infty.$$

In the following, we show a proof for rigidity assuming $\mathrm{sys}_2(M,g)=4\pi$. In this case, Σ_k has non-empty intersection with the compact set

$$K=\phi^{-1}\left(\left[-\frac{1}{8},\frac{1}{8}\right]\right).$$

We have $\mathrm{area}(\Sigma_k)\leq\mathrm{area}(\Sigma_0)$ from direct comparison. The curvature estimate and the minimizing property implies that Σ_k converges to an area-minimizing boundary Σ with multiplicity one, which has area no greater than $\mathrm{area}(\Sigma_0)$ and non-empty intersection with K. Fixed a point p in Σ, we can find point p_k in Σ_k such that $p_k\to p$ as $k\to\infty$ from the convergence. Notice that we can always normalize the function u_k in (5.3) to satisfy $u_k(p_k)=1$. As a consequence, u_k converges smoothly to a smooth positive function u on Σ due to Harnack inequality and standard elliptic estimates.

Next we prove that Σ has non-negative Ricci curvature. Once this has been done, Σ has to be compact due to the well-known fact that a non-compact complete Riemannian manifold with non-negative Ricci curvature has infinite volume (see [Yau76]). Combined with the fact $\mathrm{sys}_2(M,g)=4\pi$, a similar argument as in the proof of Theorem 1.2 then yields that the universal covering of (M,g) is isometric to the product manifold $\mathbb{S}^2\times\mathbb{R}^2$. So let us focus on a proof for the Ricci flatness of Σ. From the uniform curvature estimate and area bound for Σ_k and Σ, there is a universal constant r_0 such that

$$\Sigma_k\cap K\subset B^{\Sigma_k}_{r_0}(p_k)\quad\text{and}\quad\Sigma\cap K\subset B^{\Sigma}_{r_0}(p),$$

where $B^{\Sigma_k}_{r_0}(p_k)$ denotes the intrinsic r_0-ball of Σ_k centered at the point p_k and so is $B^{\Sigma}_{r_0}(p)$.

In the first step, we show that $(\tilde\Sigma,\tilde g)$ has non-negative Ricci curvature, where $\tilde\Sigma=\Sigma\times\mathbb{S}^1$ and $\tilde g=g_\Sigma+u^2\mathrm{d}t^2$. Actually, this comes from a careful analysis on elliptic operators

$$\mathcal{L}_k=\Delta_{\tilde g_k}+1-\frac{R_{\tilde g_k}}{2}\quad\text{and}\quad\mathcal{L}=\Delta_{\tilde g}+1-\frac{R_{\tilde g}}{2}.$$

Observe that we have $R_{\tilde g}\geq 2$ from the fact $R_{\tilde g_k}\geq 2-C\epsilon_k$. We claim that $R_{\tilde g}$ must be identical to two. Otherwise, there is a point (q,θ) in $\tilde\Sigma$ such that $R_{\tilde g}>2$ at that point. As a consequence, there will be a constant $r_1>r_0$ such that

$$\mu(\mathcal{L},B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1)>0,$$

where $\mu(\mathcal{L},B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1)$ denotes the first Neumann eigenvalue of $\mathcal{L}$ on $B^{\Sigma}_{r_1}(p)$ and we will continue to use similar notations below. This implies

$$\mu(\mathcal{L}_k,B^{\Sigma_k}_{r_1}(p_k)\times\mathbb{S}^1)>0$$

for k large enough and so we have $\mu(\mathcal{L}_k,\tilde\Sigma_k)>0$ due to the fact $R_{\tilde g_k}\geq 2$ outside $B^{\Sigma_k}_{r_0}\times\mathbb{S}^1$. Now we can find a positive smooth function $\tilde v_k$ on $\tilde\Sigma_k$ such that

$$-\Delta_{\tilde g_k}v_k+\left(\frac{R_{\tilde g_k}}{2}-1\right)v_k=\mu(\mathcal{L}_k,\tilde\Sigma_k)v_k.$$

From Fischer-Colbrie-Schoen symmetrization, we can construct a new manifold $(\hat\Sigma_k,\hat g_k)$ with $R_{\hat g_k}\geq 2+2\mu(\mathcal{L}_k,\tilde\Sigma_k)$, where $\hat\Sigma_k=\tilde\Sigma_k\times\mathbb{S}^1$ and $\hat g_k=\tilde g_k+\tilde v_k^2\mathrm{d}t^2$. This will lead to the estimate

$$\mathrm{sys}_2(M,g)\leq\frac{4\pi}{1+\mu(\mathcal{L}_k,\tilde\Sigma_k)}<4\pi$$

from the proof of Theorem 1.2, which contradicts to the fact $\operatorname{sys}_2(M,g)=4\pi$. The non-negativity of the Ricci curvature $\operatorname{Ric}_{\tilde g}$ comes from a similar argument. Actually, we have the formula

$$\begin{aligned}&\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\mu(\mathcal{L}_{h,\tau},B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1)\\ &=\operatorname{vol}_{\tilde g}(B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1)^{-1}\int_{B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1}\langle h,\operatorname{Ric}_{\tilde g}\rangle_{\tilde g}\,\mathrm{d}\mu_{\tilde g}\end{aligned}$$

for any symmetric 2-tensor h with compact support in $B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1$, where

$$\mathcal{L}_{h,\tau}=\Delta_{\tilde g-\tau h}+1-\frac{R_{\tilde g-\tau h}}{2}.$$

If the Ricci curvature $\operatorname{Ric}_{\tilde g}$ is negative at some point q, then there is a unit one form Ω in $\Lambda^1(T_q\tilde\Sigma)$ such that $\operatorname{Ric}_{\tilde g}(\omega)=\lambda\omega$ for some negative constant λ. Of course, we can extend ω to a neighborhood U of q contained in $B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1$ such that $\operatorname{Ric}_{\tilde g}(\omega,\omega)<0$ holds in U. Let η be a non-negative cut-off function with compact support in U satisfying $\eta(q)>0$. With $h=\eta\operatorname{Ric}_{\tilde g}(\omega,\omega)\omega\otimes\omega$, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0}\mu(\mathcal{L}_{h,\tau},B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1)>0$$

and so there is a positive constant τ_0 such that $\mu(\mathcal{L}_{h,\tau_0},B^{\Sigma}_{r_1}(p)\times\mathbb{S}^1)>0$. Writing Σ_k as graphs over Σ, we can view h as a symmetric 2-tensor on $\tilde\Sigma_k$. Consider the metric

$$\tilde g_{k,\tau_0}=\tilde g_k-\tau_0 h$$

and the associated operator

$$\mathcal{L}_{k,h,\tau_0}=\Delta_{\tilde g_k-\tau_0 h}+1-\frac{R_{\tilde g-\tau_0 h}}{2}.$$

It holds for k large enough that $\mu(\mathcal{L}_{k,h,\tau_0},B^{\Sigma_k}_{r_1}(p_k))>0$. Repeating Fischer-Colbrie-Schoen symmetrization argument above and using the fact $\tilde g_k\leq\tilde g_{k,\tau_0}$ as quadratic forms, we conclude that

$$\operatorname{sys}_2(M,g)\leq\frac{4\pi}{1+\mu(\mathcal{L}_{k,h,\tau_0},B^{\Sigma_k}_{r_1}(p_k))}<4\pi,$$

which leads to a contradiction again. From the proof of Theorem 1.2, deformation for lapse function u_k is allowed when we do above argument. In particular, we can obtain the estimate $\operatorname{Ric}_{\tilde g}(\partial_t,\partial_t)\equiv 0$ for the tangential vector ∂_t along $\mathbb{S}^1$.

Now, we are ready to prove the non-negativity of the Ricci curvature of Σ. From a direct computation and the fact $\operatorname{Ric}_{\tilde g}(\partial_t,\partial_t)=0$, we see that u is a positive harmonic function on Σ. Let $v=\log u$, then we have

$$\Delta v=-|\nabla v|^2.$$

Let η be a smooth non-negative cut-off function such that $\eta \equiv 1$ on $B_r^{\Sigma}(p)$, $\eta \equiv 0$ outside $B_{2r}^{\Sigma}(p)$ and $|\nabla \eta| \le 2/r$. Integral by parts, we have

$$\begin{aligned} -\int_{\Sigma} \eta^2 |\nabla v|^2 \,\mathrm{d}\sigma &= \int_{\Sigma} \eta^2 \Delta v \,\mathrm{d}\sigma \\ &= -\int_{\Sigma} 2\eta \nabla \eta \cdot \nabla v \,\mathrm{d}\sigma \\ &\ge -\frac{1}{2}\int_{\Sigma} \eta^2 |\nabla v|^2 \,\mathrm{d}\sigma - 2\int_{\Sigma} |\nabla \eta|^2 \,\mathrm{d}\sigma. \end{aligned}$$

It follows

$$\int_{B_r^{\Sigma}(p)} |\nabla v|^2 \,\mathrm{d}\sigma \le \int_{\Sigma} \eta^2 |\nabla v|^2 \,\mathrm{d}\sigma \le 4\int_{\Sigma} |\nabla \eta|^2 \,\mathrm{d}\sigma \le \frac{16}{r^2}\,\mathrm{area}(\Sigma).$$

Since Σ has finite area, we see that u is a constant function by letting $r \to +\infty$. In particular, $\tilde{\Sigma}$ is the Riemannian product of manifold Σ and $\mathbb{S}^1$ and the Ricci curvature of Σ is non-negative. □

6. Open questions

We present here some open questions.

The relation between positive scalar curvature and 2-systole in dimension 4 and more is a recent discovery and much remains unknown in this direction. As one goal of this field, we raise up with the following *homotopical 2-systole conjecture.*

Conjecture 6.1. *Let (M, g) be a closed Riemannian manifold with scalar curvature R_g no less than 2. If the universal covering of M is homtopically equivalent to $\mathbb{S}^2$, then $\mathrm{sys}_2(M, g) \le 4\pi$.*

The affirmative answer to above conjecture in dimension two and three comes quickly from Gauss-Bonnet formula and Theorem 1.4. As special cases, it follows from Theorem 1.2 the conjecture holds for $\mathbb{S}^2 \times \mathbb{T}^{n-2}$ with dimension n no greater than 7.

When the topology is less restricted, the situation is less clear. Even in dimension 4 the following quetion is open:

Question 6.2. *Let g be a metric of scalar curvature at least 4 on $\mathbb{S}^2 \times \mathbb{S}^2$, is $\mathrm{sys}_2(g) \le 4\pi$?*

Theorem 1.3 shows that this holds when the metric is streched enough, this estimate can also be shown to hold for warped products of constant curvature 2-spheres.

Beyond product of spheres, the situation has not been investigated much yet. Any progress on the following question would be worthwhile for instance:

Question 6.3. *Let g be a metric of scalar curvature at least 24 on $\mathbb{CP}^2$, is $\mathrm{sys}_2(g) \le \mathrm{sys}_2(g_{FS})$? Here g_{FS} is the Fubini-Study metric on $\mathbb{CP}^2$ with sectional curvature between 1 and 4.*

In higher dimensions, products of 2-spheres and higher dimensional complex projective spaces could also be investigated.

Another question is the applicability of hypersurface methods in dimension greater than seven. Though great efforts have been made in the last years, it is still unclear to the authors wether they can be used to extend the results of this survey in dimension 8 and more.

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