Multifractal analysis based on weak scaling exponent

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1. Introduction

Multifractal analysis based on the Hölder or p-exponent presupposes that the data belong to L^∞ or L^p (or to H^p if $p<1$). This condition is not always true. If we don't want to regularize the data by fractional integration, we can perform a multifractal analysis based on the weak scaling exponent, which does not presuppose any regularity for the data, and can be used in the context of temperate distributions. In this prospective work, we propose multi-resolution quantities adapted to this exponent in order to investigate the numerical feasibility of the method and we show its relevance for white Gaussian noise, for which no p-exponent can be used, and we apply it on the cadence of marathon runners.

2. (θ,ω)-leaders

Let ψ be a "wavelet". The wavelet coefficients of a signal X are defined by $c_{j,k}= 2^{j/2}\int_{\mathbb{R}} X(t)\psi_{j,k}(t)dt$. The wavelet scaling function η_X is defined by

$$2^{-j}\sum_k |c_{j,k}|^p \sim 2^{-\eta_X(p)j}.$$

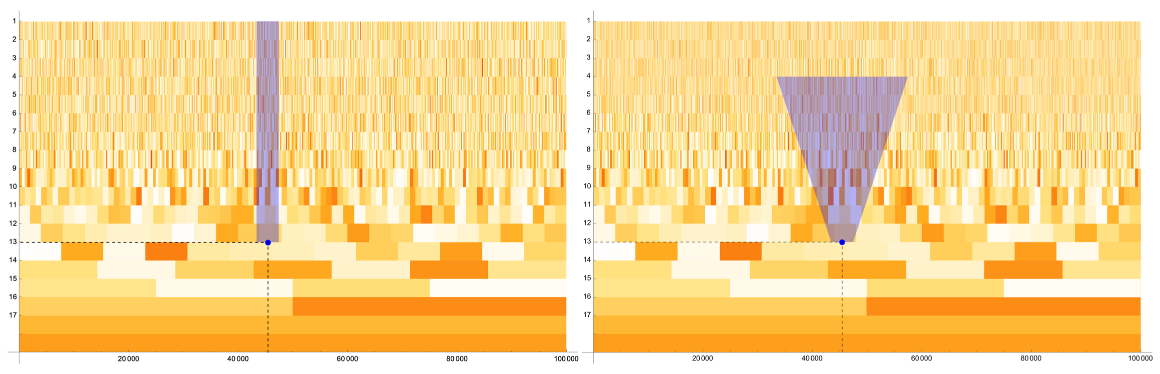
If $\eta_X(p)<0$, $\forall p>0$, then $X\notin L^p(\mathbb{R})$ and we can use the weak scaling exponent. It was introduced by Yves Meyer from the (θ,ω)-leaders.

A function $\omega:\mathbb{N}\to\mathbb{R}^+$ is *sub-exponential growth* if it is increasing and verifies $\omega(j)\to+\infty$ and $\log(\omega(j))/j\to 0$ when j tends to 0.

Let (θ,ω) a couple of sub-exponential growth function. The set $V_\omega(j,k)$ are the indices (j',k') such that

$$j\le j'\le j+\theta(j) \quad \text{and} \quad \left|\frac{k}{2^j}-\frac{k'}{2^{j'}}\right|\le\frac{\omega(j)}{2^j}.$$

The (θ,ω)-leaders are defined by $d_{j,k}=\sup_{(j',k')\in V_\omega(j,k)}|c_{j',k'}|$.



representation of the wavelet coefficients in time-scale half plane: In grey the selected wavelet coefficients for leaders (left) and (θ,ω)-leaders (right)

3. Weak scaling exponent

The weak scaling exponent is characterized by:

$$h_X^{ws}(t_0)=\liminf_{j\to+\infty}\frac{\log(d_{j,k}(t_0))}{\log(2^{-j})}.$$

where $d_{j,k}(t_0)$ is the (θ,ω)-leader associated with the dyadic interval $[k2^{-j},(k+1)2^{-j}[$ which contains t_0.

The (θ,ω)-multifractal spectrum of a function X is $D_X^{ws}:\ H\mapsto \dim_H\left(\{t\in\mathbb{R}\ :\ h_X^{ws}(t)=H\}\right)$. The scaling function ζ_X associated with (θ,ω)-leader is defined by: $2^{-j}\sum_k|d_{j,k}|^p\sim 2^{-\zeta_X(p)j}$. In the log-log plot regressions we took care of the quantification effect of the data following the results of [1].
The Legendre transform of ζ_X is $\mathcal{L}_X(H)=\inf_{p\in\mathbb{R}}(d+Hp-\zeta_X(p))$. It is used to estimate the (θ,ω)-spectrum. Indeed $D_X^{ws}(H)\le L_X(H)$.

7. Références

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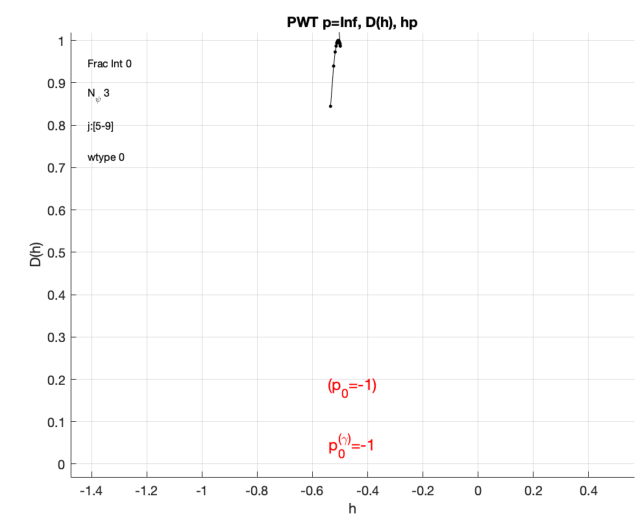
4.Gaussian white noise

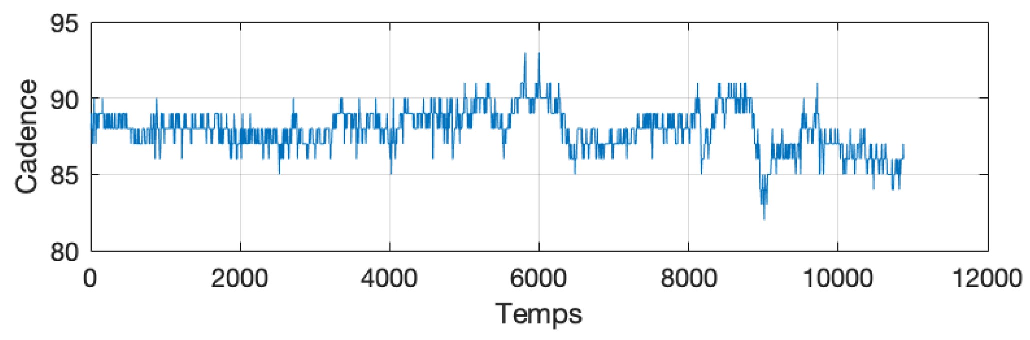
The Gaussian white noise $W(t)$ can be defined as the derivative in the sense of distrubution of the Brownian motion. No *p-exponent* is suitable to perform the multifraclat analysis and it can be shown that its weak scaling exponent satisfies:

$$a.e.\quad \forall t\in\mathbb{R},\quad h_W^{ws}(t)=-\frac{1}{2}.$$

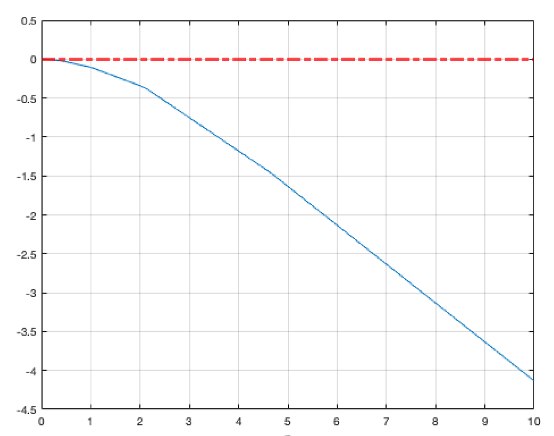


Legendre spectrum estimation of the white Gaussian noise multifractal spectrum. The (θ,ω)-leaders formalism provides an excellent estimate of the spectrum whose support is very tight around the point $h=-1/2$.

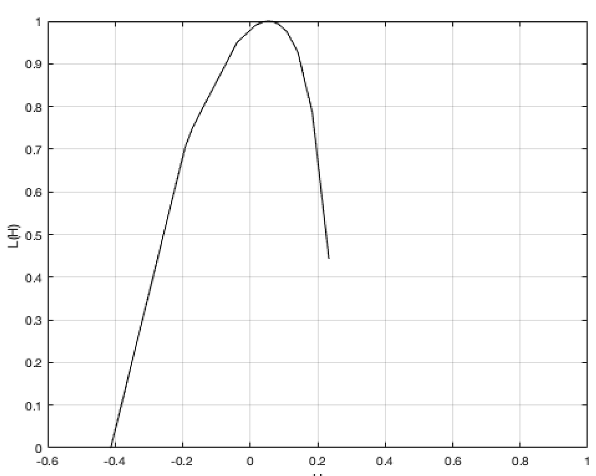
5.Multifractal analysis of cadence



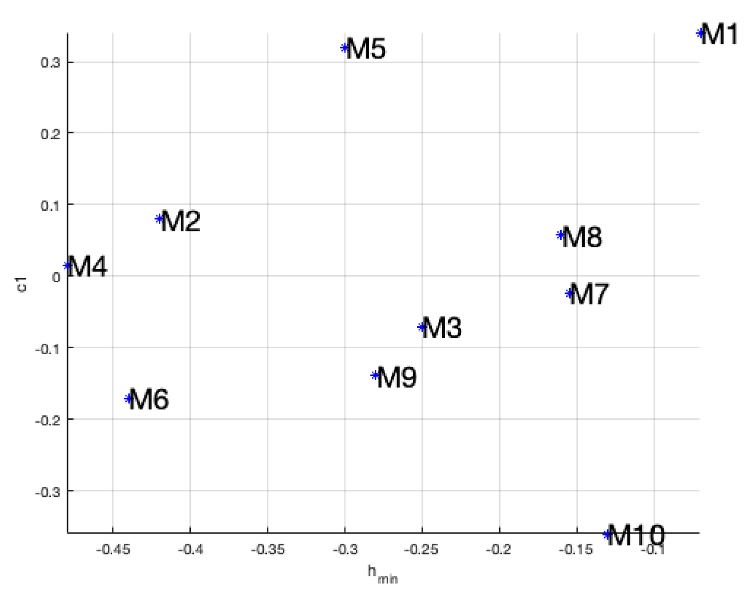
Representation of a cadence signal of a marathon runner (number of steps per minute).



Estimation of wavelet scaling function $\eta(p)$ of the cadence.
$\eta(p)<0$, for all p>0. Hence, it is not possible to apply multifractal analysis based on p-exponent.



The Legendre spectrum of the cadence using the (θ,ω)-leaders with $\theta(j)=j^{0,25}$ and $\omega(j)=j$, $j>0$. We extract two parameters from the spectrum, the uniform regularity exponent (H_{min}) and the almost everywhere exponent of the signal (c_1^{ws}).



Multifractal analysis applied to the weak scaling exponent allows to put into light differences between the behaviours of the 10 marathon runners using the representation of the couple (H_{min},c_1^{ws}), which might lead to new classification tools useful to understand in a more precise way their running caracteristics.

6. Conclusions

We have shown that an analysis based on weak scaling exponent allows us to perform a multifractal analysis directly on the data, and that it is efficient. In addition, the variability over the whole run based on cadence doesn't seem to be directly related to performance. An open question is to understand its physiological interpretation.