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Dyson models under renormalization and in weak fields

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Abstract:

We consider one-dimensional long-range spin models (usually called *Dyson models*), consisting of Ising ferromagnets with a slowly decaying long-range pair potentials of the form $\frac{1}{|i-j|^\alpha}$, mainly focusing on the range of slow decays $1 < \alpha \leq 2$. We describe two recent results, one about renormalization and one about the effect of external fields at low temperature.

The first result states that a decimated long-range Gibbs measure in one dimension becomes non-Gibbsian, in the same vein as comparable results in higher dimensions for short-range models.

The second result addresses the behaviour of such models under inhomogeneous fields, in particular external fields which decay to zero polynomially as $\frac{1}{(|i|+1)^\gamma}$. We study how the critical decay power of the field, γ, for which the phase transition persists and the decay power α of the Dyson model compare, extending recent results for short-range models on lattices and on trees. We also briefly point out some analogies between these results.

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1 Introduction

In this short review we investigate some properties of one-dimensional long-range spin models, also known as Dyson models. In his original work, Dyson [13] considered an Ising spin system with formal Hamiltonian given by

$$H(\omega) = -\sum_{i>j} J(|i-j|)\omega_i\omega_j$$

and $J(n) \geq 0$ for $n \in \mathbb{N}$ e.g. of the form $J(n) = n^{-\alpha}$.

There is no phase transition for this model, if the series $M_0 = \sum_{n=1}^{\infty} J(n)$ is infinite, since then there is an infinite energy gap between the ground states and all other states, which yields that at all finite temperatures the system is expected to be ordered. Neither is there a transition if $|\{n : J(n) \neq 0\}| < \infty$, by [49], since then the system is disordered at all finite temperatures. See also [6] and [8] for accessible proofs of different versions of this absence of a transition under conditions of sufficiently fast polynomial decay of $J(n)$. Thus in particular, there is no phase transition for $J(n)$ being of finite range, and neither for $J(n) = n^{-\alpha}$ with $\alpha > 2$.

A conjecture due to Kac and Thompson [34], early on, stated that there should be a phase transition for low enough temperatures if and only if $\alpha \in (1, 2]$. Dyson proved a part of the Kac-Thompson conjecture, namely that for long-range models of the form $n^{-\alpha}$ with $\alpha \in (1, 2)$ there is a phase transition. Note that for $M_0 < \infty$ the infinite-volume measure is well defined.

We will consider, analogously to Dyson, one-dimensional ferromagnetic models with slowly decaying pair interactions of the form $J(|i-j|) = \frac{1}{|i-j|^{\alpha}}$, for appropriate values of the decay parameter, $\alpha \in (1, 2]$, which display a phase transition at low temperature. This makes Dyson models particularly interesting, because they thus can exhibit phase coexistence even in one dimension, which is very unusual. Varying this decay parameter plays a similar role as varying the dimension in short-range models. This can be done in a continuous manner, so one obtains analogues of well-defined models in continuously varying non-integer dimensions. This is a major reason why these models have attracted a lot of attention in the study of phase transitions and critical behaviour (see e.g. [7] and references therein).

In this paper, we first sketch the proof of the fact that, at low enough temperature, under a decimation transformation the low-temperature measures of the Dyson models are mapped to non-Gibbsian measures. Indeed, similar to what happens for short-range models in higher dimensions, in the phase transition region ($1 < \alpha \leq 2$ and low enough temperature), decimating the Gibbs measures to half the spins leads to non-Gibbsianness of the decimated measures. This is obtained by showing the alternating configuration to be a point of essential discontinuity for the (finite-volume) conditional probabilities of the decimated Gibbs measures.

Just as with external fields or boundary conditions, the configuration of renormalised spins, acting on the system of "hidden spins" which are to be integrated out, can prefer one of the phases, and there are choices where this preference depends only on spins far away. The renormalised spins can act as some kind of (possibly correlated) random field, acting on the other (hidden) spins.

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We have extended our analysis to consider the effects of more general, possibly decaying, external fields on Dyson models and discuss how Dyson models in external fields decaying to zero as $\frac{1}{(|i|+1)^\gamma}$ behave as regards phase coexistence. Again similarly to what happens in short-range models, it appears that the existence of a plurality of Gibbs measures persists when the decay of the field is fast enough, whereas for slowly decaying fields we expect that there is only one Gibbs measure which survives, namely the one favoured by the field. What the appropriate decay parameter of the field, γ, which separates the two behaviours is, depends on the Dyson decay parameter α. This extends recent results on short-range models on either lattices or trees.

The review is organized as follows. In Section 2, we introduce notations and definitions of Gibbs measures and describe what is known about phase transitions in Dyson models. In Section 3, we introduce the decimation transformation – a renormalization transformation that keeps odd or even spins only – and sketch how to prove non-Gibbsianness at low temperature for the decimated Gibbs measures of the Dyson models. We show that, conditioned on the even spins to be alternating, a "hidden phase transition" occurs in the system of odd spins. In Section 4 we will discuss Dyson models in decaying fields.

2 Gibbs Measures and Dyson Models

2.1 Specifications and Measures

We refer to [17] and [5] for proofs and more details on the general formalism considered here. Dyson models are ferromagnetic Ising models with long-range pair-interactions in one dimension, possibly with an external field which we will take possibly inhomogeneous, random and/or correlated.

We study these models within a more general class of lattice (spin) models with Gibbs measures on infinite-volume product configuration spaces $(\Omega, \mathcal{F}, \rho) = (E^{\mathbb{Z}}, \mathcal{E}^{\otimes \mathbb{Z}}, \mu_o^{\otimes Z})$, the single-site state space being the Ising space $E = \{-1, +1\}$, with the a priori counting measure $\mu_0 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$. We denote by $\mathcal{S}$ the set of the finite subsets of $\mathbb{Z}$ and, for any $\Lambda \in \mathcal{S}$, write $(\Omega_\Lambda, \mathcal{F}_\Lambda, \rho_\Lambda)$ for the finite-volume configuration space $(E^\Lambda, \mathcal{E}^{\otimes \Lambda}, \mu_o^{\otimes \Lambda})$ – and extend afterwards the notations when considering infinite subsets $S \subset \mathbb{Z}$ and (restricted) infinite-volume configuration spaces $(\Omega_S, \mathcal{F}_S, \mu_S) \ni \sigma_S$.

Microscopic states or configurations, denoted by $\sigma, \omega, \eta, \tau,$, etc., are elements of Ω equipped with the product topology of the discrete topology on E for which these configurations are close when they coincide on large finite regions Λ (the larger the region, the closer). For $\omega \in \Omega$, a neighborhood base is provided by

$$\mathcal{N}_L(\omega) = \left\{ \sigma \in \Omega : \sigma_{\Lambda_L} = \omega_{\Lambda_L}, \ \sigma_{\Lambda_L^c} \text{ arbitrary} \right\}, \ L \in \mathbb{N}, \ \Lambda_L := [-L, +L] \in \mathcal{S}.$$

For any integers $N > L$, we shall also consider particular open subsets of neighborhoods

$$\mathcal{N}_{N,L}^+(\omega) = \left\{ \sigma \in \mathcal{N}_L(\omega) : \sigma_{\Lambda_N \setminus \Lambda_L} = +_{\Lambda_N \setminus \Lambda_L}, \ \sigma \text{ arbitrary otherwise} \right\} \left(\text{resp. } \mathcal{N}_{N,L}^-(\omega) \right).$$

We denote by $C(\Omega)$ the set of continuous (quasilocal) functions on Ω, characterized by

$$f \in C(\Omega) \iff \lim_{\Lambda \uparrow \mathcal{S}} \sup_{\sigma, \omega : \sigma_\Lambda = \omega_\Lambda} | f(\omega) - f(\sigma) | = 0. \tag{2.1}$$

Monotonicity for functions and measures concerns the natural partial (FKG) order "$\leq$", which we have on our Ising spin systems : $\sigma \leq \omega$ if and only if $\sigma_i \leq \omega_i$ for all $i \in \mathbb{Z}$. Its maximal and minimal elements are the configurations $+$ and $-$, and this order extends to functions: $f : \Omega \longrightarrow \mathbb{R}$ is called *monotone increasing* when $\sigma \leq \omega$ implies $f(\sigma) \leq f(\omega)$. For measures, we write $\mu \leq \nu$ if and only if $\mu[f] \leq \nu[f]$ for all f monotone increasing[1].

Macroscopic states are represented by probability measures on $(\Omega, \mathcal{F}, \rho)$, whose main description – at least in mathematical statistical mechanics – is in terms of consistent systems of regular versions of finite-volume conditional probabilities with prescribed boundary conditions, within the so-called *DLR formalism* [11, 37]. To do so, one introduces families of probability kernels that are natural candidates to represent such versions of conditional probabilities.

Definition 1 (Specification) :
A specification $\gamma = (\gamma_\Lambda)_{\Lambda \in \mathcal{S}}$ on $(\Omega, \mathcal{F})$ is a family of probability kernels $\gamma_\Lambda : \Omega_\Lambda \times \mathcal{F}_{\Lambda^c} \longrightarrow [0,1]$; $(\omega, A) \longmapsto \gamma_\Lambda(A \mid \omega)$ s.t. for all $\Lambda \in \mathcal{S}$:

1. *(Properness) For all $\omega \in \Omega$, $\gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$ when $B \in \mathcal{F}_{\Lambda^c}$.*

2. *(Finite-volume consistency) For all $\Lambda \subset \Lambda' \in \mathcal{S}$, $\gamma_{\Lambda'} \gamma_\Lambda = \gamma_{\Lambda'}$ where*

$$\forall A \in \mathcal{F}, \ \forall \omega \in \Omega, \ (\gamma_{\Lambda'} \gamma_\Lambda)(A|\omega) = \int_\Omega \gamma_\Lambda(A|\sigma) \gamma_{\Lambda'}(d\sigma|\omega). \tag{2.2}$$

These kernels also act on functions and on measures: for all $f \in C(\Omega)$ or $\mu \in \mathcal{M}_1^+$,

$$\gamma_\Lambda f(\omega) := \int_\Omega f(\sigma) \gamma_\Lambda(d\sigma|\omega) = \gamma_\Lambda[f|\omega] \text{ and } \mu \gamma_\Lambda[f] := \int_\Omega (\gamma_\Lambda f)(\omega) d\mu(\omega) = \int_\Omega \gamma_\Lambda[f|\omega] \mu(d\omega).$$

Definition 2 (DLR measures) :
A probability measure μ on $(\Omega, \mathcal{F})$ is said to be consistent with a specification γ (or specified by γ) when for all $A \in \mathcal{F}$ and $\Lambda \in \mathcal{S}$

$$\mu[A|\mathcal{F}_{\Lambda^c}](\omega) = \gamma_\Lambda(A|\omega), \ \mu-\text{a.e. } \omega. \tag{2.3}$$

We denote by $\mathcal{G}(\gamma)$ the set of measures consistent with γ.

A *specification* γ is said to be **quasilocal** when for any local function f, the image $\gamma_\Lambda f$ should be a continuous function of the boundary condition :

$$\gamma \text{ quasilocal} \iff \gamma_\Lambda f \in C(\Omega) \text{ for any } f \text{ local (or any } f \text{ in } C(\Omega)). \tag{2.4}$$

A *measure* is said to be quasilocal when it is specified by a quasilocal specification.

A particularly important subclass of quasilocal measures consists of the *Gibbs measures* with (formal) Hamiltonian H defined via a potential Φ, a family $\Phi = (\Phi_A)_{A \in \mathcal{S}}$ of local functions $\Phi_A \in \mathcal{F}_A$. The contributions of spins in finite sets A to the total energy define the *finite-volume Hamiltonians with free boundary conditions*

$$\forall \Lambda \in \mathcal{S}, \ H_\Lambda(\omega) = \sum_{A \subset \Lambda} \Phi_A(\omega), \ \forall \omega \in \Omega. \tag{2.5}$$

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[1]We denote $\mu[f]$ for the expectation $\mathbb{E}_\mu[f]$ under a measure μ.

To define Gibbs measures, we require for Φ that it is *Uniformly Absolutely Convergent* (UAC), i.e. that $\sum_{A\ni i} \sup_\omega |\Phi_A(\omega)| < \infty, \forall i \in \mathbb{Z}$. One then can give sense to the *Hamiltonian at volume* $\Lambda \in \mathcal{S}$ *with boundary condition* ω defined for all $\sigma, \omega \in \Omega$ as $H_\Lambda^\Phi(\sigma|\omega) := \sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\sigma_\Lambda\omega_{\Lambda^c})(< \infty)$. The *Gibbs specification at inverse temperature* $\beta > 0$ is then defined by

$$\gamma_\Lambda^{\beta\Phi}(\sigma \mid \omega) = \frac{1}{Z_\Lambda^{\beta\Phi}(\omega)} \, e^{-\beta H_\Lambda^\Phi(\sigma|\omega)}(\rho_\Lambda \otimes \delta_{\omega_{\Lambda^c}})(d\sigma) \tag{2.6}$$

where the partition function $Z_\Lambda^{\beta\Phi}(\omega)$ is an important normalizing constant. Due to the, in fact rather strong, UAC condition, these specifications are quasilocal. It appears that the converse is also true up to a non-nullness condition[2] (see e.g. [26, 18, 36, 52, 42]) and one can take :

Definition 3 (Gibbs measures) :
$\mu \in \mathcal{M}_1^+$ *is a Gibbs measure iff* $\mu \in \mathcal{G}(\gamma)$, *where* γ *is a non-null and quasilocal specification.*

Quasilocality, called *Almost Markovianness* in [52], is a natural way to extend the global (two-sided) Markov property. When $\mu \in \mathcal{G}(\gamma)$ is quasilocal, then for any f local and $\Lambda \in \mathcal{S}$, the conditional expectations of f w.r.t. the outside of Λ are μ-a.s. given by $\gamma_\Lambda f$, by (2.2), and each conditional probability has a version which itself is a continuous function of the boundary condition, so one gets for any ω

$$\lim_{\Delta\uparrow\mathbb{Z}} \sup_{\omega^1,\omega^2\in\Omega} \left| \mu\big[f|\mathcal{F}_{\Lambda^c}\big](\omega_\Delta\omega_{\Delta^c}^1) - \mu\big[f|\mathcal{F}_{\Lambda^c}\big](\omega_\Delta\omega_{\Delta^c}^2) \right| = 0 \tag{2.7}$$

Thus, for Gibbs measures the conditional probabilities always have continuous versions, or equivalently there is no point of essential discontinuity. Those are configurations which are points of discontinuity for ALL versions of the conditional probability. In particular one cannot make conditional probabilities continuous by redefining them on a measure-zero set if such points exist. In the generalized Gibbsian framework, one also says that such a configuration is a *bad configuration* for the considered measure, see e.g. [42]. The existence of such bad configurations implies non-Gibbsianness of the associated measures.

2.2 Dyson models: Ferromagnets in One Dimension

Definition 4 (Dyson models) :
Let $\beta > 0$ *be the inverse temperature and consider* $1 < \alpha \leq 2$. *We call a* Dyson model with decay parameter α *the Gibbs specification (2.6) with pair-potential* Φ^D *defined for all* $\omega \in \Omega$ *by*

$$\Phi_A^D(\omega) = -\frac{1}{|i-j|^\alpha}\omega_i\omega_j \text{ when } A = \{i,j\} \subset \mathbb{Z}, \text{ and } \Phi_A^D \equiv 0 \text{ otherwise.} \tag{2.8}$$

We shall also consider Dyson models with non-zero magnetic fields $h = (h_i)_{i\in\mathbb{Z}}$ *acting as an extra self-interaction part* $\Phi_A^D(\omega) = -h_i\omega_i$ *when* $A = \{i\} \subset \mathbb{Z}$

We first use that as a consequence of FKG property [22, 30], the Dyson specification is *monotonicity-preserving* [3], which implies that the weak limits obtained by using as boundary

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[2]expressing that $\forall \Lambda \in \mathcal{S}$, $\forall A \in \mathcal{F}_\Lambda$, $\rho(A) > 0$ implies that $\gamma_\Lambda(A|\omega) > 0$ for any $\omega \in \Omega$.
[3]in the sense that for all bounded increasing functions f, and $\Lambda \in \mathcal{S}$, the function $\gamma_\Lambda^D f$ is increasing.

conditions the maximal and minimal elements of the order $\leq$ are well defined and are the extremal elements of $\mathcal{G}(\gamma^D)$.

Proposition 1 *[19, 30, 38]: For $\alpha > 1$ (and not only for $\alpha \in (1,2]$), the weak limits*

$$\mu^-(\cdot) := \lim_{\Lambda \uparrow \mathbb{Z}} \gamma_\Lambda^D(\cdot|-) \text{ and } \mu^+(\cdot) := \lim_{\Lambda \uparrow \mathbb{Z}} \gamma_\Lambda^D(\cdot|+) \tag{2.9}$$

are well-defined, translation-invariant and extremal elements of $\mathcal{G}(\gamma^D)$. For any f bounded increasing, any other measure $\mu \in \mathcal{G}(\gamma^D)$ satisfies

$$\mu^-[f] \leq \mu[f] \leq \mu^+[f]. \tag{2.10}$$

Moreover, μ^- and μ^+ are respectively left-continuous and right-continuous.

While μ^- and μ^+ coincide at high temperatures, and at all temperatures when there is fast decay, $\alpha > 2$, one main peculiarity of this one-dimensional model is thus that when the range is long enough ($1 < \alpha \leq 2$), it is possible to recover low-temperature behaviours usually associated to higher dimensions for the standard Ising model, in particular phase transitions can occur. For more details on the history of the proofs, one can consult [17] and references therein or below.

Proposition 2 *[14, 24, 48, 26, 25, 23, 1, 7, 44, 33].*

The Dyson model with potential (2.8), for $1 < \alpha \leq 2$, exhibits a phase transition at low temperature:

$$\exists \beta_c^D > 0, \text{ such that } \beta > \beta_c^D \implies \mu^- \neq \mu^+ \text{ and } \mathcal{G}(\gamma^D) = [\mu^-, \mu^+]$$

where the extremal measures μ^+ and μ^- are translation-invariant. They have in particular opposite magnetisations $\mu^+[\sigma_0] = -\mu^-[\sigma_0] = M_0(\beta, \alpha) > 0$ at low temperature. Moreover, the Dyson model in a non-zero homogeneous field h has a unique Gibbs measure.

We remark that the infinite-volume limit of a state (or a magnetisation) in which there is a $+$ (resp. $-$)-measure or a Dyson model in a field $h > 0$ (resp. $h < 0$) outside some interval is the same as that obtained from $+$ (resp. $-$)-boundary conditions (independent of the magnitude of h). This can be e.g. seen by an extension of the arguments of [40], see also [39]. Notice that taking the $+$-measure of the zero-field Dyson model outside a finite volume enforces this same measure inside (even before taking the limit); adding a field makes it it more positive, and taking the thermodynamic limit then recovers the same measure again.

The case of $\alpha = 2$ is more complicated to analyse, and richer in its behaviour, than the other ones. There exists a hybrid transition (the "Thouless effect"), as the magnetisation is discontinuous while the energy density is continuous at the transition point. Moreover, there is second transition below this transition temperature. In the intermediate phase there is a positive magnetisation with non-summable covariance, while at very low temperatures the covariance decays at the same rate as the interaction, which is summable. For these results, see [1, 31, 32], and also the more recent description in [44].

3   Decimation

We first apply a decimation transformation to the lattice $2\mathbb{Z}$. Similarly to what was discussed in [15], to analyse whether the transformed measure is a Gibbs measure, and in particular to show that it is non-Gibbsian, we have to show that conditioned on a particular configuration of the transformed spins, the "hidden spins" display a phase transition. If we choose this particular configuration to be the alternating one, each hidden spin feels opposite terms from the left and the right side, coming from all odd distances. Thus the conditioned model is a Dyson model in zero field, at a reduced temperature. As such it has phase transition.

To translate this hidden phase transition into nonlocality of the "visible" transformed spins follows straightforwardly the arguments of [15]. See [17] for the details. We make use of the fact that one can define global specifications, so there are no measurability problems due to global conditioning.

We start from μ^+ (2.9), the +-phase of a Dyson model without external field in the phase transition region and apply the *decimation transformation*

$$T \colon (\Omega, \mathcal{F}) \longrightarrow (\Omega', \mathcal{F}') = (\Omega, \mathcal{F}); \ \omega \ \longmapsto \omega' = (\omega_i')_{i \in \mathbb{Z}}, \ \text{with } \omega_i' = \omega_{2i} \qquad (3.11)$$

Denote $\nu^+ := T\mu^+$ the decimated +-phase, formally defined as an image measure via

$$\forall A' \in \mathcal{F}', \ \nu^+(A') = \mu^+(T^{-1}A') = \mu^+(A) \text{ where } A = T^{-1}A' = \left\{\omega : \omega' = T(\omega) \in A'\right\}.$$

We study the continuity of conditional expectations under decimated Dyson Gibbs measures of the spin at the origin when the outside is fixed in some special configuration ω_{alt}'. By definition,

$$\nu^+[\sigma_0' | \mathcal{F}_{\{0\}^c}](\omega') = \mu^+[\sigma_0 | \mathcal{F}_{S^c}](\omega), \ \nu^+-\text{a.s.} \qquad (3.12)$$

where $S^c = (2\mathbb{Z}) \cap \{0\}^c$, i.e. with $S = (2\mathbb{Z})^c \cup \{0\}$ is not finite: *the conditioning is not on the complement of a finite set*, and although the extension of the DLR equation to infinite sets is direct in case of uniqueness of the DLR-measure for a given specification [19, 21, 27], it can be more problematic otherwise: it is valid for finite sets only and measurability problems might arise in case of phase transitions when one wants to extend them to infinite sets. Nevertheless, beyond the uniqueness case, such an extension was made possible by Fernández and Pfister [19] in the case of attractive models. As we will make essential use of it, we describe it now in our particular case. The concept they introduced is that of a *global specification*, and this is in fact a central tool in some of our arguments.

Definition 5 (Global specification [19]) *:*
A global specification Γ *on* $\mathbb{Z}$ *is a family of probability kernels* $\Gamma = (\Gamma_S)_{S \subset \mathbb{Z}}$ *on* $(\Omega_S, \mathcal{F}_{S^c})$ *such that for* any *S subset of* $\mathbb{Z}$:

1. $\Gamma_S(\cdot|\omega)$ *is a probability measure on* $(\Omega, \mathcal{F})$ *for all* $\omega \in \Omega$.

2. $\Gamma_S(A|\cdot)$ *is* $\mathcal{F}_{S^c}$*-measurable for all* $A \in \mathcal{F}$.

3. $\Gamma_S(B|\omega) = \mathbf{1}_B(\omega)$ *when* $B \in \mathcal{F}_{S^c}$.

4. *For all* $S_1 \subset S_2 \subset \mathbb{Z}$, $\Gamma_{S_2}\Gamma_{S_1} = \Gamma_{S_2}$ *where the product of kernels is made as in (2.2).*

Similarly to the consistency with a (local) specification, one introduces the *compatibility of measures with a global specification*.

Definition 6 *Let* Γ *be a global specification. We write* $\mu \in \mathcal{G}(\Gamma)$*, or say that* $\mu \in \mathcal{M}_1^+$ *is* Γ-compatible*, if for all* $A \in \mathcal{F}$ *and any* $S \subset \mathbb{Z}$*,*

$$\mu[A|\mathcal{F}_{S^c}](\omega) = \Gamma_S(A|\omega), \ \mu-a.e. \ \omega. \tag{3.13}$$

Note, by considering $S = \mathbb{Z}$, that $\mathcal{G}(\Gamma)$ contains at most one element.

In the case considered here, we get a global specification Γ^+ such that $\mu^+ \in \mathcal{G}(\Gamma^+)$, with $S = (2\mathbb{Z})^c \cup \{0\}$ consisting of the *odd integers plus the origin*. Hence $S = (2\mathbb{Z})^c \cup \{0\}$ and (3.12) yields for ν^+-a.e. all $\omega' \in \mathcal{N}_{\Lambda'}(\omega'_{\text{alt}})$ and $\omega \in T^{-1}\{\omega'\}$:

$$\nu^+[\sigma'_0|\mathcal{F}_{\{0\}^c}](\omega') = \Gamma_S^+[\sigma_0|\omega] \ \mu^+-a.e.(\omega). \tag{3.14}$$

to eventually get (see [19, 17]) an expression of the latter in terms of a constrained measure $\mu_{(2\mathbb{Z})^c \cup \{0\}}^{+,\omega}$, with $\omega \in T^{-1}\{\omega'\}$ so that we get for any $\omega' \in \mathcal{N}_{\Lambda'}(\omega'_{\text{alt}})$,

$$\nu^+[\sigma'_0|\mathcal{F}_{\{0\}^c}](\omega') = \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+,\omega} \otimes \delta_{\omega_{2\mathbb{Z} \cap \{0\}^c}}[\sigma_0].$$

Thanks to monotonicity-preservation, the constrained measure is explicitly built as the weak limit obtained by +-boundary conditions fixed after a freezing the constraint to be ω on the even sites :

$$\forall \omega' \in \mathcal{N}_{\Lambda'}(\omega'_{\text{alt}}), \forall \omega \in T^{-1}\{\omega'\}, \ \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+,\omega}(\cdot) = \lim_{I \in \mathcal{S}, I \uparrow (2\mathbb{Z})^c \cup \{0\}} \gamma_I^D(\cdot \mid +_{(2\mathbb{Z})^c \cup \{0\}}) \omega_{2\mathbb{Z} \cap \{0\}^c}). \tag{3.15}$$

Observe that when a phase transition holds for the Dyson specification – at low enough T for $1 < \alpha \leq 2$ – the same is true for the constrained specification *with alternating constraint* (although at a lower T). This phase transition then implies non-Gibbsianness of ν^+ (and for all other Gibbs measures of the model, see [17]).

Theorem 1 *([17]) : Let* $\alpha \in (1,2]$*, let* μ *be a Gibbs measure for the interaction given by (2.8) and let the transformation* T *be defined by (3.11). Then for low temperatures,* $\beta > 2^\alpha \beta_c^D$*, the decimated measure* $\nu = T \circ \mu$ *is non-Gibbs.*

Sketch of Proof: The main idea is to prove that the alternating configuration is an essential point of discontinuity for the decimated conditional expectations. As already observed, because any non-fixed site at all odd distances has a positive and a negative spin whose influences cancel, conditioning by this alternating configuration yields a constrained model that is again a Dyson model at zero field, but at a temperature which is higher by 2^α. This again has a low-temperature transition in our range of decays $1 < \alpha \leq 2$. The coupling constants are indeed multiplied by a factor $2^{-\alpha}$, due to only even distances occurring between interacting (hidden) spins.

To prove non-Gibbsianness in [17], we essentially follow the proof strategy sketched in [15], by showing that within a neighborhood $\mathcal{N}_L(\omega'_{\text{alt}})$, there exists two subneighborhoods $\mathcal{N}_{N,L}^\pm$ of positive measure on which the conditional magnetizations defined on $\mathcal{N}_N, L^\pm$

$$M^+ = M^+(\omega) = \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+,\omega^+}[\sigma_0] \text{ and } M^- = M^-(\omega) = \mu_{(2\mathbb{Z})^c \cup \{0\}}^{+,\omega^-}[\sigma_0]. \tag{3.16}$$

differ significantly.

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The role of the "annulus" where configurations are constrained to be either $+$ or $-$ is played by two large intervals $[-N, -L-1]$ and $[L+1, N]$. Due to the long range of the interaction, their might be a direct influence from the boundary beyond the annulus, to the central interval. To avoid effects from this influence, we take N *much* larger than L. An argument based on "equivalence of boundary conditions" as in e.g. [6] , under a choice $N = L^{\frac{1}{\alpha-1}}$ then implies that (3.16) does hardly depend on ω.

Once this choice of big annulus is made, observe that if we constrain the spins in these two intervals to be either $+$ or $-$, within these two intervals the measures on the unfixed spins are close to those of the Dyson-type model in a positive, c.q. negative, magnetic field. As those measures are unique Gibbs measures, no influence from the boundary can be transmitted.

Indeed, in contrast to the case of the purely alternating configuration, in the case when we condition on all primed spins to be $+$ (resp. $-$) in these large annuli, there is no phase transition and the system of unprimed spins has a unique Gibbs measure. It is a Dyson model, again at a heightened temperature, but now in a homogeneous external field, with positive (resp. negative) magnetisation $+M_0(\beta, \alpha) > 0$ (resp. $-M_0(\beta, \alpha) < 0$), stochastically larger (resp. smaller) than the zero-field $+$ (resp. $-$)-measure.

In the $-$-case, in the annulus the magnetisation of the -even-distance- Dyson-Ising model is essentially that of the model with a negative homogeneous external field $-h$ everywhere, which at low enough temperature and for L large enough is close to (and in fact smaller than) the magnetisation of the Dyson-Ising model under the zero-field $-$-measure, i.e to $-M_0(\beta, \alpha) < 0$. Thus the inner interval where the constraint is alternating feels a $-$-like condition from outside its boundary. On the other hand, the magnetisation with the constraint ω^+ will be close to or bigger than $+M_0(\beta, \alpha) > 0$ so that a non-zero difference is created at low enough temperature. One needs again to adjust the sizes of L and N to be sure that boundary effects from outside the annulus are negligible in the inner interval.

Thus, for a given $\delta > 0$, e.g. $\delta = \frac{1}{2}M_0(\beta, \alpha)$, for arbitrary L one can find $N(L)$ large enough, such that the expectation of the spin at the origin differs by more than δ. One can therefore feel the influence from the decimated spins in the far-away annulus, however large the central interval of decimated alternating spins is chosen. Thus, indeed it holds that $M^+ - M^- > \delta$, uniformly in L. $\qquad\square$

In our choice of decimated lattice we made use of the fact that the constrained system, due to cancellations, again formed a zero-field Dyson-like model. This does not work for decimations to more dilute lattices, but although the original proofs of Dyson [13] and of Fröhlich and Spencer [24], or the Reflection Positivity proof of [23] do no longer apply to such periodic-field cases, the contour-like arguments of [7] and [33] could presumably still be modified to include such cases. Compare also [35].

The analysis of [9] which proves existence of a phase transition for Dyson models in random magnetic fields for a certain interval of α-values should imply that in that case there are many more, random, configurations which all are points of discontinuity. We note that choosing independent spins as a constraint provides a random field which is correlated. However, these correlations decay enough that this need actually not spoil the argument. Similarly, one should be able to prove that decimation of Dyson models in a weak external field will result in a non-Gibbsian measure. An interesting question would be to perform the analysis of [45]) or [43] to get a.s. configuration-dependent correlation decays.

On the other side of the Gibbs-non-Gibbs analysis, when the range of the interaction is

lower, i.e. for $\alpha > 2$, or the temperature is too high, uniqueness holds, for all possible constraints and the transformed measures should be Gibbsian. Some standard high-temperature results apply, which were already discussed in [15].

About these shorter-range models, (i.e. long-range models with faster polynomial decay), Redig and Wang [46] have proved that Gibbsianness was conserved, providing in some cases ($\alpha > 3$) a decay of correlation for the transformed potential. In our longer-range models, for intermediate temperatures (below the transition temperature but above the transition temperature of the alternating-configuration-constrained model) decimating, both +- and −-measures, should imply Gibbsianness, essentially due to the arguments as proposed for short-range models in [29].

4 Dyson models in decaying fields

In this section we consider one-dimensional Dyson models in a decaying field with decay parameter γ. The corresponding interaction $\Phi_A^D(\omega)$ is defined by

$$\Phi_A^D(\omega) = \begin{cases} -\frac{J}{|i-j|^\alpha}\omega_i\omega_j & \text{if } A = \{i,j\} \\ -\frac{h}{(|i|+1)^\gamma}\omega_i & \text{if } A = \{i\} \end{cases} \tag{4.17}$$

for some $J, h > 0$. The question raised in [5] is whether it is possible to extend results from e.g. [7, 9], and in particular to investigate whether and under which conditions the existence of two distinct phases prevails in the presence of an external field. Let us mention that one-dimensional Dyson models in a field were considered before, for example in [35], where uniqueness was proven for fields which are either strong enough ($\Phi_{\{i\}}^D = h_i\omega_i$, where there exists $h_0 > 0$ such that $|h_i| > h_0$) or periodic in large enough blocks.

The main tool we will use are the one-dimensional contours of [7]. Recall that in [7] the authors prove that for $\alpha \in (\alpha^*, 2]$ where $\alpha^* := 3 - \frac{\log(3)}{\log(2)}$ and $h = 0$ there exists $\beta_{c,0}^D > 0$ such that for all $\beta > \beta_{c,0}^D$

$$M_0(\beta, \alpha) = \mu^+[\sigma_0] = -\mu^-[\sigma_0] > 0$$

i.e. there is spontaneous magnetization yielding non-uniqueness of the Gibbs measures, $\mu^+ \neq \mu^-$. This result was generalized to all values of $\alpha \in (1, 2]$ in [44].

Phase coexistence in a positive external field is an unusual phenomenon, since typically Gibbs measures for models in a field are unique. It was previously observed in nearest neighbour pair potentials with polynomially decaying fields in $d \geq 2$, see [2, 3, 10] or for sufficiently fast decaying (but not necessarily summable) fields on trees [4]. In [3] it is proven that in nearest-neighbour models for $\gamma > 1$ and low enough temperatures, there are multiple Gibbs states, whereas for $\gamma < 1$ there is a unique one.

Pirogov-Sinai is a robust and often applicable version of the Peierls contour argument, applicable in $d \geq 2$, which is the most generally applicable approach in higher dimensions. In [7], inspired by and extending results of the seminal paper of Fröhlich and Spencer [24], the authors presented a contour argument which works even for long-range models in one dimension, in particular, it worked for one-dimensional Dyson models with $\alpha^* < \alpha \leq 2$. The techniques used in [7] rely on developing a graphical representation of spin configurations in terms of triangular *contours*.

It turns out that for a one-dimensional long-range model in a decaying field, depending on the relation between α and γ, there can be either one or two extremal Gibbs measures. We can prove the following theorem.

Theorem 2 ([5]) *Let $\alpha \in (1, 2]$ and $\gamma > \max\{\alpha - 1, \alpha^* - 1\}$ be the exponents of the Dyson model w.r.t. an interaction Φ^D given by (4.17). Then, there exists $\beta_{c,h}^D > 0$ s.t. for all $\beta > \beta_{c,h}^D$ we have $M_0(\beta, \alpha, \gamma) > 0$, i.e. $\mu^+ \neq \mu^-$.*

Sketch of proof: The main idea of the proof is to extend the analysis in [7, 9], combined with [44]. Consider a finite-volume Gibbs measure on an interval, say $\Lambda = [-N, N]$ and fix +-boundary conditions. Each spin configuration σ can be uniquely mapped into a triangle configuration $\underline{T} = (T_1, ..., T_n)$ where endpoints of the triangles are defined by interface points dividing plus from minus spins. Contours Γ are collections of triangles T_i such that they are in some sense *well separated* from each other and subadditive, so that we obtain a lower bound for the energy of given triangle configuration $\underline{T}$. Phase coexistence will follow from the well-known Peierls argument for $d > 1$, i.e. fromthe estimate that for for β sufficiently large

$$\mu_\Lambda^+[\sigma_o = -1] \leq \mu_\Lambda^+[\{o \in \Gamma\}] \leq \frac{1}{Z_\Lambda^+} \sum_{\Gamma \ni o} \sum_{\Gamma \text{ compatible}} e^{-\beta H(\underline{T})} < \frac{1}{2}.$$

The main difficulty then is to obtain a *good* energetic lower bound for the Hamiltonian including the effect of the external field. $\qquad\square$

Physically, an argument explaining the statement of the theorem goes as follows: There is a competition between the effect of the pair interaction and that of the external field. Having minus boundary conditions means that inserting a large interval $[-L, L]$ of plus spins will cost an energy of order

$$\sum_{|i|<L} \sum_{|j|>L} |i - j|^{-\alpha} = O(L^{2-\alpha}).$$

However, the gain in energy due to the spins following the external magnetic field is of order

$$\sum_{|i|<L} |i|^{-\gamma} = O(L^{1-\gamma}).$$

Thus, (somewhat similar to an Imry-Ma argument), we see that for $\gamma > \alpha - 1$ we should expect that the field is too weak to overcome the boundary conditions and the plus and minus measures are different: $\mu^+ \neq \mu^-$.

When the opposite case pertains, that is $\gamma < \alpha - 1$, there should be a unique Gibbs measure, with a magnetisation in the direction of the field, whatever the boundary conditions employed. We are in the process of rigorising this picture.

In fact, the analogous prediction in the 2-dimensional short-range model has been fully proved by [3, 10], also giving that the critical value for γ equals 1, where is possible to prove the phase transition even in the critical case, assuming that h is small enough. Here we have the same situation, we can extend the theorem above for the case when $\gamma = \max\{\alpha - 1, \alpha^* - 1\}$ if we take h small enough.

The restriction on γ involving α^* seems due to technical reasons, since we use arguments developed in [7]. Also, again based on their paper, for technically reasons we require the nearest-neighbor term to be strong enough.

However, from the physical argument sketched above, we expect that these limitations should

not be required and the argument should work, just assuming the inequality between γ and α .

References

[1] M. Aizenman, J. Chayes, L. Chayes, C. Newman. Discontinuity of the Magnetization in the One-Dimensional $1/\mid x-y\mid^2$ Percolation, Ising and Potts Models. *J. Stat. Phys.* **50**, no 1/2:1-40, 1988.

[2] R. Bissacot, L. Cioletti. Phase transition in ferromagnetic Ising models with non-uniform external fields. *J. Stat. Phys.* **139**, no 5: 598-617, 2010.

[3] R. Bissacot, M. Cassandro, L. Cioletti, E. Presutti. Phase transition in ferromagnetic Ising models with spatially dependent magnetic fields. *Comm. Math. Phys.* **337**, 41-53, 2015.

[4] R. Bissacot, E.O. Endo , A.C.D. van Enter. Stability of the phase transition of critical-field Ising model on Cayley trees under inhomogeneous external fields, arXiv 1611.00424.

[5] R. Bissacot, E.O. Endo , A.C.D. van Enter, B. Kimura, W.M. Ruszel. Phase transition of long-range Dyson models in inhomogeneous fields, Manuscript in preparation.

[6] J. Bricmont, J. Lebowitz, C.-E. Pfister. On the Equivalence of Boundary Conditions. *J. Stat. Phys.* **21**, No 5: 573–582, 1979.

[7] M. Cassandro, P.A. Ferrari, I. Merola, E. Presutti. Geometry of Contours and Peierls Estimates in $d=1$ Ising Models with Long Range Interactions. *J. Math. Phys.* **46**(5), 0533305, 2005.

[8] M. Cassandro, E. Olivieri. Renormalization group and analyticity in one dimension: A proof of Dobrushin's theorem. *Comm. Math. Phys.* **80**:255–269, 1981.

[9] M. Cassandro, E. Orlandi, P. Picco. Phase Transition in the 1D Random Field Ising Model with Long Range Interaction. *Comm. Math. Phys.* **288**:731–744, 2009.

[10] L. Cioletti, R. Vila. Graphical representations for Ising and Potts models in general external fields. *J. Stat. Phys.* **162**: 81–122, 2016.

[11] R.L. Dobrushin. The Description of a Random Field by Means of Conditional Probabilities and Conditions of its Regularity. *Theo. Proba. Appl.* **13**:197–224, 1968.

[12] E.B. Dynkin. Sufficient statistics and extreme points. *Ann. Proba.* **6**, No. 5:705–730, 1978.

[13] F.J. Dyson. Existence of a Phase Transition in a One-Dimensional Ising Ferromagnet. *Comm. Math. Phys. 12*:91–107, 1969.

[14] F.J. Dyson. An Ising ferromagnet with discontinuous long-range order. *Comm. Math. Phys. 21*:269–283, 1971.

[15] A.C.D. van Enter, R. Fernández, A.D. Sokal. Regularity Properties and Pathologies of Position-Space R.G. Transformations: Scope and Limitations of Gibbsian Theory. *J. Stat. Phys.* **72**:879-1167, 1993.

[16] A.C.D. van Enter, C. Külske. Two connections between random systems and non-Gibbsian measures. *J. Stat. Phys.* **126**: 1007–1024, 2007.

[17] A.C.D. van Enter, A. Le Ny. Decimation of the Dyson-Ising Ferromagnet. arXiv 1603.05409, 2016.

[18] R. Fernández. Gibbsianness and non-Gibbsianness in Lattice random fields. In *Mathematical Statistical Physics. Proceedings of the 83rd Les Houches Summer School (july 2005)*, Elsevier, A. Bovier, A.C.D. van Enter, F. den Hollander, F. Dunlop eds., 2006.

[19] R. Fernández, C.-E. Pfister. Global specifications and non-quasilocality of projections of Gibbs measures. *Ann. Proba.* **25**, no 3:1284-315, 1997.

[20] H. Föllmer. Phase Transition and Martin Boundary. In *Séminaires de Probabilités IX, Université de Strasbourg*. Lecture Notes in Mathematics **465**:305–317. Berlin-Heidelberg-New York, Springer, 1975.

[21] H. Föllmer. On the Global Markov Property. In *Quantum fields: Algebras, Processes* (L. Streit, ed.):293-302, Springer, New York, 1980.

[22] C.M. Fortuin, P.W. Kasteleyn, J. Ginibre. Correlation Inequalities on some Partially Ordered Sets. *Comm. Math. Phys.* **22**:89–103, 1971.

[23] J. Fröhlich, R.B. Israel, E.H. Lieb, B. Simon. Phase Transitions and Reflection Positivity. I. General Theory and Long Range Lattice Models. *Comm. Math. Phys.* **62**: 1-34, 1978.

[24] J. Fröhlich, T. Spencer. The Phase Transition in the One-Dimensional Ising Model with $1/r^2$ interaction energy. *Comm. Math. Phys.* **84**:87–101, 1982.

[25] S. Friedli, Y. Velenik. Equilibrium Statistical Mechanics : a Concrete Mathematical Introduction. http://www.unige.ch/math/folks/velenik/smbook/index.html

[26] H.O. Georgii. *Gibbs Measures and Phase Transitions*. De Gruyter Studies in Mathematics, Vol 9, Berlin–New York, 1988, 2nd Edition 2011.

[27] S. Goldstein. Remarks on the Global Markov Property. *Comm. Math. Phys.* **74**:223-234, 1980.

[28] R.B. Griffiths. Peierls Proof of Spontaneous Magnetization in a Two-Dimensional Ising Ferromagnet. *Phys. Rev.* **2**, 136:A437–A439, 1964.

[29] K. Haller, T. Kennedy. Absence of renormalization pathologies near the critical temperature. Two examples. *J.Stat. Phys.* **85**:607–637, 1996.

[30] P. Hulse. On the Ergodic Properties of Gibbs States for Attractive Specifications. *J. London Math. Soc.* (2) **43**, no 1:119–124, 1991.

[31] J. Imbrie. Decay of Correlations in One-Dimensional Ising Model with $J_{ij} =\mid i - j \mid^{-2}$. *Comm. Math. Phys.* **85**:491–515, 1982.

[32] J. Imbrie, C. M. Newman. An Intermediate Phase with Slow Decay of Correlations in One-Dimensional $1/ \mid x - y \mid^2$ Percolation, Ising and Potts Models. *Comm. Math. Phys.* **118**:303–336, 1988.

[33] K. Johansson. Condensation of a One-Dimensional Lattice Gas. *Comm. Math. Phys.* **141**: 41–61, 1991.

[34] M. Kac, C.J. Thompson. Critical behaviour of several lattice models with long-range interaction. *Rockefeller University*, 1968.

[35] A. Kerimov. A block effect of external field in the one-dimensional ferromagnetic Ising model with long-range interaction. *J.Phys. A: Math. Theor.* **40**: 40407-40414, 2007.

[36] O. Kozlov. Gibbs Description of a System of Random Variables. *Problems Inform. Transmission.* **10**:258–265, 1974.

[37] O.E. Lanford, D. Ruelle. Observables at Infinity and States with Short Range Correlations in Statistical Mechanics. *Comm. Math. Phys.* **13**:194–215, 1969.

[38] J.L. Lebowitz. Coexistence of Phases for Ising Ferromagnet. *J. Stat. Phys.* **16**, no 6:463–476, 1977.

[39] J.L. Lebowitz. Thermodynamic Limit of the Free Energy and Correlation Functions of Spins Systems. *Acta Phys. Austr.*, Suppl XVI:201-220, 1976.

[40] J.L. Lebowitz, O. Penrose. Analytic and clustering properties of thermodynamic functions and distribution functions for classical lattice and continuum systems. *Comm. Math. Phys.* **11**:99124, 1968.

[41] T.D. Lee, C.N Yang. Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model. *Physical Review* **87**:410–419, 1952.

[42] A. Le Ny. *Introduction to Generalized Gibbs measures.* Ensaios Matemáticos **15**, 2008.

[43] A. Le Ny. Almost Gibbsianness and Parsimonious Description of the Decimated 2d-Ising model. *J. Stat. Phys.* **152**, no. 2:305–335, 2013.

[44] J. Littin, P. Picco. Quasiadditive estimates on the Hamiltonian for the One-dimensional Long-Range Ising Model. Preprint, 2016.

[45] C. Maes, F. Redig, S. Shlosman, A. Van Moffaert. Percolation, Path Large deviations and Weak Gibbsianity.*Comm. Math. Phys* **209**, no 8:517-545, 1999.

[46] F. Redig, F. Wang. Transformations of One-Dimensional Gibbs Measures with Infinite Range Interaction. *Markov Proc. Relat. Fields* **16**, issue 4, 2010.

[47] D. Ruelle. Statistical Mechanics of a One Dimensional Lattice Gas. *Comm. Math. Phys.* **9**:267–278, 1968.

[48] D. Ruelle. On the use of "small external fields" in the problem of symmetry breakdown in statistical mechanics. *Ann. Phys.* **69**:364–374, 1972.

[49] G. Rushbrooke, H. Ursell. On one-dimensional regular assemblies. *Proc. Cambridge Phil. Soc. 44*, **263**, 1948.

[50] B. Simon, A.D. Sokal. Rigorous Entropy-Energy Arguments. *J. Stat. Phys.* **25**, no 4:679–694, 1981.

[51] A.D. Sokal. Existence of Compatible Families of Proper Regular Conditional Probabilities. *Z. Wahrsch. verw. Geb.* **56**:537-548, 1981.

[52] W.G. Sullivan. Potentials for Almost Markovian Random Fields. *Comm. Math. Phys.* **33**:61-74, 1976.